Algebra Definitions, Rules, and Theorems

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Properties of Real Numbers:

A1 closure $\forall\, a, b \in \mathbb{R}, a + b \in \mathbb{R}$
A2 commutativity $\forall\, a, b \in \mathbb{R}, a + b = b + a$
A3 associativity $\forall\, a, b, c \in \mathbb{R}, (a + b) + c = a + (b + c)$
A4 additive identity $\exists!\, 0\ such\ that\ \forall\, a \in \mathbb{R}, a + 0 = 0 + a = a$
A5 additive inverse $\forall\, a \in \mathbb{R}, \exists!\, (-a) such\ that\ a + (-a) = (-a) + a = 0$

M1 closure $\forall\, a, b \in \mathbb{R}, ab \in \mathbb{R}$
M2 commutativity $\forall\, a, b \in \mathbb{R}, ab = ba$
M3 associativity $\forall\, a, b, c \in \mathbb{R}, (ab)c = a(bc)$
M4 multiplicative identity $\exists!\, 1\ such\ that\ \forall\, a \in \mathbb{R}, a \cdot 1 = 1 \cdot a = a$
M5 multiplicative inverse $\forall\, a \in \mathbb{R}\ s.t.\ a \neq 0, \exists!\ \frac{1}{a}\ such\ that\ a \cdot \frac{1}{a} = \frac{1}{a} \cdot a = 1$

D distributive $\forall\, a, b, c \in \mathbb{R}, a(b + c) = ab + ac$

Properties of Equality:

Let $a, b, c \in \mathbb{R}$.
reflexive $a = a$
symmetric $if\ a = b, then\ b = a$
transitive $if\ a = b\ and\ b = c, then\ a = c$
substitution $if\ a = b, then\ a\ may\ be\ replaced\ by\ b\ in\ any\ expression\ that\ involves\ a$

Properties of Fractions:

$\forall\ \frac{a}{b}\ and\ \frac{c}{d},\ a, b, c, d \in \mathbb{R}, and\ b, d \neq 0,$
equality $\frac{a}{b} = \frac{c}{d}\ iff\ ad = bc$ (cross-multiplication)
equivalent fractions $\frac{a}{b} = \frac{ac}{bc}, c \neq 0$
addition $\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$
subtraction $\frac{a}{b} - \frac{c}{b} = \frac{a-c}{b}$
multiplication $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$
division $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}, c \neq 0$
sign $-\frac{a}{b} = \frac{-a}{b} = \frac{a}{-b}$

Absolute Value:

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

$\forall\, a, b \in \mathbb{R}$, the distance between a & b is $|a - b| = |b - a|$.

Properties of Exponents:

Let $p, q, r \in \mathbb{Q}, \quad a, b \in \mathbb{R}, \quad a, b > 0$

Product	$b^p \cdot b^q = b^{p+q}$
Quotient	$\frac{b^p}{b^q} = b^{p-q}$
Power	$(b^p)^q = b^{pq}$
Distributive	$(a^p b^q)^r = a^{pr} b^{qr}$, $\left(\frac{a^p}{b^q}\right)^r = \frac{a^{pr}}{b^{qr}}$
Reciprocal	$b^{-p} = \frac{1}{b^p}$

Properties of Radicals:

Let $m, n \in \mathbb{N}$ *and* $a, b \in \mathbb{R}$ *such that* $a^{\frac{1}{n}}, b^{\frac{1}{n}} \in \mathbb{R}$.

$$\sqrt[n]{b} = b^{\frac{1}{n}}$$
$$\left(\sqrt[n]{b}\right)^m = \sqrt[n]{b^m} = b^{\frac{m}{n}}$$
$$\sqrt[n]{b^n} = \begin{cases} b, & if\ n\ is\ odd \\ |b|, & if\ n\ is\ even \end{cases}$$

$$\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$$
$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$$
$$\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$$

Polynomials:

<u>monomial</u> – a constant, variable, or product of a constant and one or more variables with nonnegative integer exponents
<u>coefficient</u> – the constant part of a monomial
<u>degree of a monomial</u> – the sum of the exponents of the variables
<u>polynomial</u> – sum of a finite number of monomials; each monomial is a <u>term</u> of the polynomial
<u>degree of a polynomial</u> – largest degree of the terms in the polynomial
<u>binomial</u> – simplified polynomial with 2 terms
<u>trinomial</u> – simplified polynomial with 3 terms
<u>general form</u> of an nth degree polynomial is $a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2x^2 + a_1x + a_0x$
$a_n x^n$ is the <u>lead term</u>
a_n is the <u>leading coefficient</u>
a_0 is the <u>constant term</u>
<u>Special forms:</u>
$(x + y)(x + y) = x^2 + 2xy + y^2$
$(x - y)(x - y) = x^2 - 2xy + y^2$
$(x - y)(x + y) = x^2 - y^2$

Factoring:

<u>Factorization theorem:</u> The trinomial $ax^2 + bx + c$, with integer coefficients a, b, c, can be factored as the product of two binomials with integer coefficients if and only if $b^2 - 4ac$ (the discriminant) is a perfect square
<u>Special forms:</u>
$x^2 + 2xy + y^2 = (x + y)(x + y)$
$x^2 - 2xy + y^2 = (x - y)(x - y)$
$x^2 - y^2 = (x - y)(x + y)$
$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$
$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$

Rational Expressions:

A rational expression is a fraction in which the numerator and denominator are polynomials.
The domain of a rational expression is the set of all real numbers that can be used as replacements for the variable, i.e. all real numbers except for those that cause the denominator to be 0.
The Zero Product Property states: $If\ AB = 0, then\ A = 0\ or\ B = 0.$
The same properties hold as for fractions.
A complex fraction is a fraction in which the numerator or denominator contains one or more fractions.

Linear Equations:

A linear equation is an equation of the form $ax + b = 0\ ,\ a \neq 0$
contradiction – equation with no solution
conditional equation – equation true for only certain values of x
identity – equation that is true for all values of x (in the domain)

Absolute Value Equations:
$For\ a \epsilon \mathbb{R}, a \neq 0, \quad |f(x)| = a \quad if\ and\ only\ if \quad f(x) = a\ or\ f(x) = -a$

Quadratic Equations:

Standard form: $ax^2 + bx + c = 0,\ \ a \neq 0$
Zero Product Property: $If\ AB = 0,\ then\ A = 0\ or\ B = 0$
Square Root Theorem: $If\ A^2 = B,\ then\ A = \pm\sqrt{B}$

Steps for Completing the Square:
1. Move the constant term to the right-hand side.

$$ax^2 + bx = -c$$

2. Factor out the x^2 coefficient from all terms on the left-hand side.

$$a(x^2 + \frac{b}{a}x) = -c$$

3. Divide both sides by the x^2 coefficient.

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

4. Complete the square by taking half of the x coefficient, squaring it, and adding it to both sides.

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$

5. Rewrite the left-hand side as a perfect square and simplify the right-hand side.

$$\left(x + \frac{b}{2a}\right)^2 = \frac{-4ac+b^2}{4a^2}$$ (Note: this may not look "simplified" but with actual numbers, it will; just write as single fraction)

6. Apply square root theorem.

$$x + \frac{b}{2a} = \pm\sqrt{\frac{-4ac+b^2}{4a^2}}$$

7. Solve for x and simplify.

$$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Quadratic Formula: $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

Complex Numbers:

Standard form: $a + bi$, where a is the real part and b is the imaginary part

$i = \sqrt{-1}$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$

$i^a = i^{4b+r} = (i^4)^b i^r = i^r, where\ r\ is\ the\ remainder\ (0, 1, 2\ or\ 3)\ when\ a\ is\ divided\ by\ 4$

$a + bi$ and $a - bi$ are complex conjugates; their product $(a + bi)(a - bi)$ is the real number $a^2 + b^2$

To add complex numbers $a + bi$ and $c + di$, just combine like terms: $(a + c) + (b + d)i$

To multiply complex numbers $(a + bi)(c + di)$, FOIL, combine like terms, and replace i^2 with -1.

To divide complex numbers, multiply numerator and denominator by the conjugate of the denominator.

$\frac{a+bi}{c+di} \cdot \frac{c-di}{c-di}$

Caution: to multiply $\sqrt{a}\sqrt{b}$ when $a, b < 0$, write in terms of i before multiplying.

Classify roots (solutions) using the discriminant:

The discriminant is the $b^2 - 4ac$ part of the quadratic formula.

If $b^2 - 4ac > 0$, the quadratic equation will have two distinct real roots (solutions).

If $b^2 - 4ac = 0$, the quadratic equation will have one real “double” root.

If $b^2 - 4ac < 0$, the quadratic equation will have two complex conjugate roots.

Properties of Inequalities:

Addition: adding the same real number to both sides of an inequality preserves the direction of the inequality symbol. $e.g.,\quad a < b\ \&\ a + c < b + c\ are\ equivalent\ for\ a, b, c \epsilon \mathbb{R}$

Multiplication: multiplying both sides by the same positive real number preserves the direction, but multiplying both sides by the same negative real number changes the direction of the inequality. $e.g.,\ a < b\ \&\ ac > bc\ are\ equivalent\ if\ c < 0.$

Compound Inequalities:

If two inequalities are joined with AND, take the intersection of the two solution sets.

If two inequalities are joined with OR, take the union of the two solution sets.

Absolute Value Inequalities:

For any variable expression $f(x)$ and any nonnegative real number a,

$|f(x)| \le a\quad if\ and\ only\ if\quad f(x) \le a\ and\ f(x) \ge -a,\qquad i.e. -a \le f(x) \le a$

remember – less thAND

$|f(x)| \ge a\quad if\ and\ only\ if\quad f(x) \ge a\ or\ f(x) \le -a$

remember - greatOR

Distance and Midpoint:

The distance between two real numbers a and b is $d = |a - b|$.

The distance between two points in the plane, $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

The midpoint between two points in the plane, $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

Basic Graphing:

The graph of an equation is the set of all points (x, y) whose coordinates satisfy the equation.
Intercepts are points with zero x- or y-intercepts (where the graph intersects the axis). To find the y-intercept, set $x = 0$ and solve for y. To find the x-intercept, set $y = 0$ and solve for x.

Circles:

A circle is the set of points in the plane that are a fixed distance away from a specified point. The distance is the radius of the circle and the point is the center.
The standard form of the equation of a circle with center (h, k) and radius r is

$$(x-h)^2 + (y-k)^2 = r^2$$

Given the general form of the equation of a circle, one can find the center and radius by completing the square:
1. Given an equation for a circle in the form $x^2 + y^2 + ax + by + c = 0$,
2. Move the constant to the right hand side and rearrange terms so that like variables are grouped.

$$x^2 + ax + y^2 + by = c$$

3. Then, complete the square once for the x's and again for the y's, remembering to add the new constants to both sides of the equation.

$$x^2 + ax + \left(\frac{a}{2}\right)^2 + y^2 + by + \left(\frac{b}{2}\right)^2 = c + \left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2$$

4. Rewrite the left hand side as the sum of two perfect squares and combine the right hand side.

$$\left(x + \frac{a}{2}\right)^2 + \left(y + \frac{b}{2}\right)^2 = c + \left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2$$

5. Here, our center is $\left(-\frac{a}{2}, -\frac{b}{2}\right)$ and the radius is $\sqrt{c + \left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2}$. Note that the coordinates of the center have negative signs because the standard form equation is written with minus signs.

Functions:

Given two sets A and B, associate with each element x in A exactly one element of b, denoted $f(x)$. f is a function from A to B. A function can be thought of as the set of ordered pairs $(x, f(x))$.
The domain of a function f is the set of all x-values that make sense when plugged into the function. For example, any x-value that results in a zero in the denominator or a negative under the radical is not in the domain.
The range of a function is the set of all outputs of the domain. Specifically, the range is the set of all y-values such that $y = f(x)$ for some x in the domain.
In the equation $y = f(x)$, x is the independent variable because we can choose any x value (that makes sense) and plug it into the function, and y is the dependent variable, because the value of y is dependent on the x-value that we chose.
The vertical line test for a function states that a vertical line drawn through the graph of a function can only intersect the graph at a single point.
A function f is increasing on an interval if $f(a) < f(b)$ whenever $a < b$.
A function f is decreasing on an interval if $f(a) > f(b)$ whenever $a < b$.
A function f is constant on an interval if $f(a) = f(b)$ for all a and b.
A function f is one-to-one (1-1) if $f(a) = f(b)$ implies that $a = b$. One-to-one functions must pass the horizontal line test in addition to the vertical line test. To show that a function is 1-1, plug two different variables into the function, and set the two functions equal to each other; manipulate algebraically until you get the two variables equal to each other.
To show that a function is not 1-1, plug two different real numbers into the function that yield the same y-value.

Linear Functions:

A linear function is one of the form $f(x) = mx + b$, where m is the slope of the line and b is the y-intercept. $y = mx + b$ is called the slope-intercept form of the equation of a line.

The slope of a linear function can be found by taking the ratio of change in y-values over the change in x-values. $m = \frac{y_2 - y_1}{x_2 - x_1} =$ " $\frac{rise}{run}$ "

Given the slope m and a point (x_1, y_1) on a line, the slop-intercept form can be easily found by plugging these values into the point-slope equation: $y - y_1 = m(x - x_1)$.

Lines with a 0-slope are called horizontal lines and are of the form $y = k$ for some constant k. Vertical lines are said to have "no slope" and are of the form $x = k$.

Two lines in a plane are parallel if they never intersect. Two lines are perpendicular if their intersection forms a 90° angle.

Let l_1 be the graph of $f_1(x) = m_1x + b_1$ and let l_2 be the graph of $f_2(x) = m_2x + b_2$. l_1 and l_2 are parallel if $m_1 = m_2$. This is denoted $l_1 \parallel l_2$. l_1 and l_2 are perpendicular if $m_1 = -\frac{1}{m_2}$. This is denoted $l_1 \perp l_2$.

Quadratic Functions:

The standard form of a quadratic function is $f(x) = ax^2 + bx + c$. The graph of $f(x)$ is a parabola.

The vertex of a parabola is the lowest point on a parabola that opens up or the highest point on a parabola that opens down.

A parabola is symmetric with respect to the vertical line through its vertex. This line is called the axis of symmetry.

A more useful form of a quadratic function is $f(x) = a(x - h)^2 + k$. The graph of f is a parabola with vertex (h, k). The parabola opens up if a >0 and down if $a < 0$. The vertical line $x = h$ is the axis of symmetry.

The useful form can be gotten from the standard form by completing the square:

Starting with $f(x) = ax^2 + bx + c$,

1. Factor the x^2-coefficient out of the x^2 and x terms only.

$$f(x) = a\left(x^2 + \frac{b}{a}x\right) + c$$

2. Complete the square inside the parentheses.

$$f(x) = a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2\right) + c$$

3. Add the appropriate constant outside to cancel out the constant we added inside. Adding $\left(\frac{b}{2a}\right)^2$ inside the function is really adding $a \cdot \left(\frac{b}{2a}\right)^2$, since everything inside parentheses is being multiplied by a. Since the only thing we can add to one side of an equation without changing it is 0, we add $-a \cdot \left(\frac{b}{2a}\right)^2 = -\frac{b^2}{4a}$.

$$f(x) = a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2\right) + c - \frac{b^2}{4a}$$

4. Rewrite the parentheses as a perfect square and combine constants on the outside into a single integer or fraction.

$$f(x) = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$$

Symmetry:

The graph of an equation is <u>symmetric with respect to the y-axis</u> if replacement of x with $-x$ leaves the equation unchanged.
The graph of an equation is <u>symmetric with respect to the x-axis</u> if replacement of y with $-y$ leaves the equation unchanged.
The graph of an equation is <u>symmetric with respect to the origin</u> if replacement of x and y with $-x$ and $-y$ leaves the equation unchanged.

Even and Odd Functions:

A function f is <u>even</u> if $f(-x) = f(x)$. Even functions are symmetric with respect to the y-axis.
A function f is <u>odd</u> if $f(-x) = -f(x)$. Odd functions are symmetric with respect to the origin.

Algebra of Functions:

For all values of x for which both $f(x)$ and $g(x)$ are defined, we define the following functions:
Sum: $(f + g)(x) = f(x) + g(x)$
Difference: $(f - g)(x) = f(x) - g(x)$
Product: $(fg)(x) = f(x) \cdot g(x)$
Quotient: $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, g(x) \neq 0$

The domains of $f + g$, $f - g$, $f \cdot g$ are the intersections of the domains of f and g.
The domain of f/g is the intersection of the domains of f and g except for x such that $g(x) = 0$.

Composition of Functions:

$(g \circ f)(x) = g[f(x)]$ for all x in the domain of f such that $f(x)$ is in the domain of g

Difference Quotient:

$$\frac{f(x+h) - f(x)}{h}, h \neq 0$$

Describes how a function changes in value as the independent variable changes

Inverse Functions:

If f is a one-to-one function with domain X and range Y, and g is a function with domain Y and range X, then g is the <u>inverse function</u> of f if and only if

$f\big(g(x)\big) = x$, $\forall x \epsilon$ domain of g
$g\big(f(x)\big) = x$, $\forall x \epsilon$ domain of f

Inverse functions "undo" each other.
We often denote the inverse function of f by f^{-1}. Note that $f^{-1}(x) \neq \frac{1}{f(x)}$
How to <u>find the equation for f^{-1}</u>:
1. Substitute y for $f(x)$
2. Interchange x and y
3. Solve for y in terms of x
4. Substitute $f^{-1}(x)$ for y
5. Verify that the domain of f is the range of f^{-1}, and the range of f is the domain of f^{-1}

The graph of a function f^{-1} and the graph of its inverse function f are symmetric with respect to the line $y = x$.

Systems of Linear Equations in Two Variables:

A solution of a system of equations in two variables is an ordered pair (x, y) that is a solution of each equation. Graphically, the solution is the set of points where the lines intersect. The graphs of two linear equations can intersect at a single point, be the same line (intersect at all points), or be parallel (never intersect).

How to identify a system with no solution: If both equations can be rewritten in the form $y = mx + b$, and it is clear that the lines have the same slope but different y-intercepts, then they are parallel and the system has no solution. Alternately, you can attempt to solve the system by either substitution or elimination, and if you arrive at a contradiction ($2 = 3$, etc.), then the system has no solution.

How to identify a system with infinite solutions: If both equations can be rewritten in the form $y = mx + b$, and it is clear that the two lines are the same, then there are infinitely many solutions. Alternately, you can attempt to solve the system by either substitution or elimination, and if you arrive at an identity ($2 = 2$, etc.), then the system has infinitely many solutions. The form of the solutions is $(x, mx + b)$ if you solved for y in terms of x, or if you solved for x in terms of y, the solutions would be of the form $(f(y), y)$.

Substitution method for solving: Rearrange one of the equations to solve for one variable, and then substitute the expression into the other equation to get one equation with one variable. Solve for that variable and then plug into one of the original equations to solve for the other variable.

Elimination method for solving: The goal is to produce an equivalent system of equations that is easy to solve (i.e. one of the equations only has one variable). Things we are allowed to do:
1. Interchange any two equations
2. Replace an equation with a non-zero multiple of that equation
3. Replace an equation with the sum of that equation and a nonzero multiple of another equation

Systems of Linear Equations in More than Two Variables:

An equation of the form $Ax + By + Cz = D$ is a linear equation in three variables.
A solution of an equation in three variables is an ordered triple (x, y, z).
The graph of a linear equation in three variables is a plane.
For a linear system of equations in three variables to have a solution, the graphs of the three planes must all intersect either at a single point, along a common line, or all graphs must be the same.
The same rules apply as for solving a linear system in 2 variables by elimination. Here, the goal is to produce an equivalent system of equations in triangular form:

$$\begin{cases} Ax + By + Cz = D \\ \qquad\quad Ey + Fz = G \\ \qquad\qquad\quad Hz = K \end{cases}$$

Conic Sections

A parabola is the set of points in the plane that are equidistant from a fixed line (called the directrix) and a fixed point (called the focus) not on the directrix

Parabola with axis of symmetry the y-axis and vertex at the origin
standard form: $x^2 = 4py$
focus: $(p, 0)$
directrix: $y = -p$
Parabola with axis of symmetry the x-axis and vertex at the origin
standard form: $y^2 = 4px$
focus: $(0, p)$
directrix: $x = -p$

An ellipse is the set of all points in the plane, the sum of whose distances from two fixed points (called the foci) is a positive constant.

Ellipse with major axis on x-axis and center at the origin
standard form: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\ ,\ a > b$
length of major axis: $2a$
length of minor axis: $2b$
coordinates of vertices: $(a, 0), (-a, 0)$
coordinates of foci: $(c, 0), (-c, 0)$, where $c^2 = a^2 - b^2$
Ellipse with major axis on y-axis and center at the origin
standard form: $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1\ ,\ a > b$
length of major axis: $2a$
length of minor axis: $2b$
coordinates of vertices: $(0, a), (0, -a)$
coordinates of foci: $(0, c), (0, -c)$, where $c^2 = a^2 - b^2$

Eccentricity of an ellipse: $= \frac{c}{a}$, where c is the distance from the center to a focus and a is one-half the length of the major axis.
Because $< a\ ,\ 0 < e < 1$.
When ≈ 0 , the graph is almost a circle.
When ≈ 1 , the graph is long and thin.

A hyperbola is the set of all points in the plane, the difference between whose distances from two fixed points (Called foci) is a positive constant.

Hyperbola with transverse axis on x-axis and center at the origin
standard form: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
vertices: $(a, 0), (-a, 0)$
foci: $(c, 0), (-c, 0)\ , where\ c^2 = a^2 + b^2$
asymptotes: $y = \frac{b}{a}x\ \ and\ y = -\frac{b}{a}x$
Hyperbola with transverse axis on y-axis and center at the origin
standard form: $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$
vertices: $(0, a), (0, -a)$
foci: $(0, c), (0, -c)\ , where\ c^2 = a^2 + b^2$
asymptotes: $y = \frac{a}{b}x\ \ and\ y = -\frac{a}{b}x$

Eccentricity of a hyperbola: $= \frac{c}{a}$, where c is the distance from the center to a focus and a is the distance from the center to a vertex.