Differential Calculus – Definitions, Rules and Theorems

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Precalculus Review

Functions, Domain and Range

$f: X \to Y$ a function f from X to Y assigns to each $x \in X$ a unique $y \in Y$
the domain of f is the set X - the set of all real numbers for which the function is defined
Y is the image of x under f, denoted $f(x)$
the range of f is the subset of Y consisting of all images of numbers in X
x is the independent variable, y is the dependent variable
f is one-to-one if to each y-value in the range there corresponds exactly one x-value in the domain
f is onto if its range consists of all of Y

Implicit v. Explicit Form

$3x + 4y = 8$ implicitly defines y in terms of x
$y = -\frac{3}{4}x + 2$ explicit form

Graphs of Functions

A function must pass the vertical line test
A one-to-one function must also pass the horizontal line test
Given the graph of a basic function $y = f(x)$, the graph of the transformed function,
$y = af(bx + c) + d = af\left[b\left(x + \frac{c}{b}\right)\right] + d$, can be found using the following rules:
a - vertical stretch (mult. y-values by a)
b - horizontal stretch (divide x-values by b)
$\frac{c}{b}$ – horizontal shift (shift left if $\frac{c}{b} > 0$, right if $\frac{c}{b} < 0$)
d - vertical shift (shift up if $d > 0$, down if $d < 0$)

You should know the basic graphs of:
$y = x,\ y = x^2,\ y = x^3,\ y = |x|,\ y = \sqrt{x},\ y = \sqrt[3]{x},\ y = \frac{1}{x},$
$y = \sin x,\ y = \cos x,\ y = \tan x,\ y = \csc x,\ y = \sec x,\ y = \cot x$

For further discussion of graphing, see Precal and Trig Guides to Graphing

Elementary Functions

Algebraic functions (polynomial, radical, rational) - functions that can be expressed as a finite number of sums, differences, multiples, quotients, & radicals involving x^n
Functions that are not algebraic are trancendental (eg. trigonometric, exponential, and logarithmic functions)

Polynomial Functions

$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$

a_n = leading coefficient

$a_n x^n$ = lead term

n = degree of polynomial (largest exponent)

a_0 = constant term (term with no x)

Lead term test for end behavior: even functions behave like $y = x^2$, odd functions behave like $y = x^3$

Odd and Even Functions

$y = f(x)$ is <u>even</u> if $f(-x) = f(x)$

$y = f(x)$ is <u>odd</u> if $f(-x) = -f(x)$

Rational Functions

$$f(x) = \frac{p(x)}{q(x)}\ , \qquad p, q - polynomials, \qquad q(x) \neq 0$$

Zeros of a function

Solutions to the equation $f(x) = 0$;

When written in the form $(x, 0)$ are called the <u>x-intercepts</u> of the function

Y-intercepts

Found by evaluating $f(0)$ (plugging 0 in for x)

Vertical Asymptotes

Found by setting the denominator equal to 0 and solving for x.

Note that any factors found in both numerator and denominator will result in holes rather than vertical asymptotes.

The graph will approach but never cross a vertical asymptote.

Horizontal/Oblique Asymptotes

If degree of the numerator is smaller than the degree of denominator, there will be a horizontal asymptote at $y = 0$.

If the degree of the numerator is the same as the degree of the denominator, there will be a horizontal asymptote at $y = c$, where c is the constant ratio of the leading coefficients.

If the degree of the numerator is one higher than the degree of the denominator, there will be an oblique asymptote (one degree higher - linear, 2 degrees higher, quadratic, etc), found by long division.

Limits

Informal Definition of a Limit

If $f(x)$ becomes arbitrarily close to a single number L as x approaches c from either side, the limit of $f(x)$, as x approaches c, is L.

Note: the existence or nonexistence of $f(x)$ at $x = c$ has no bearing on the existence of the limit of $f(x)$ as x approaches c.

Formal Definition of a Limit

Let f be a function defined on an open interval containing c (except possibly at c) and let L be a real number. The statement $\lim_{x\to c} f(x) = L$ means that for each $\varepsilon > 0$, there exists a $\delta > 0$ such that if $0 < |x - c| < \delta$, then $|f(x) - L| < \varepsilon$.

Translation: If given any arbitrarily small positive number ε, there exists another small positive number δ such that $f(x)$ is ε-close to L whenever x is δ-close to c, then the limit of f as x approaches c exists and is equal to L.

Note: if the limit of a function exists, then it is unique.

Example proof of a limit using the $\boldsymbol{\varepsilon - \delta}$ definition

Problem: Prove that $\lim_{x\to -3} 2x + 5 = -1$

Proof: We want to show that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $|(2x + 5) - (-1)| < \varepsilon$ whenever $|x - (-3)| < \delta$. Since we can rewrite $|2x + 5 + 1| = |2x + 6| = 2|x + 3|$, if we take $\delta = \frac{\varepsilon}{2}$, then whenever $|x - (-3)| < \delta$, we have $|(2x + 5) - (-1)| = 2|x + 3| < 2 \cdot \frac{\varepsilon}{2} = \varepsilon$. Since we have found a δ that works for any ε, we have proved that the limit of the function is -1.

Evaluating Limits Analytically

Basic Limits

$Let\ \ b, c \in \mathbb{R}, \quad n > 0\ an\ integer, \quad f, g - functions, \quad \lim_{x\to c} f(x) = L, \quad \lim_{x\to c} g(x) = K$

1. Constant $\lim_{x\to c} b = b$
2. Identity $\lim_{x\to c} x = c$
3. Polynomial $\lim_{x\to c} x^n = c^n$
4. Scalar Multiple $\lim_{x\to c} [bf(x)] = bL$
5. Sum or Difference $\lim_{x\to c} [f(x) \pm g(x)] = L \pm K$
6. Product $\lim_{x\to c} [f(x)g(x)] = LK$
7. Quotient $\lim_{x\to c} \left[\frac{f(x)}{g(x)}\right] = \frac{L}{K} \quad , \quad K \neq 0$
8. Power $\lim_{x\to c} [f(x)]^n = L^n$

Note: If substitution yields $\frac{0}{0}$, an indeterminate form, the expression must be rewritten in order to evaluate the limit.

Examples:

1. $\lim_{x\to -1} \frac{2x^2-x-3}{x+1} = \lim_{x\to -1} \frac{(x+1)(2x-3)}{x+1} = \lim_{x\to -1}(2x-3) = \boxed{-5}$
2. $\lim_{x\to 0} \frac{\sqrt{2+x}-\sqrt{2}}{x} = \lim_{x\to 0} \frac{\sqrt{2+x}-\sqrt{2}}{x} \cdot \frac{\sqrt{2+x}+\sqrt{2}}{\sqrt{2+x}+\sqrt{2}} = \lim_{x\to 0} \frac{2+x-2}{x(\sqrt{2+x}+\sqrt{2})} = \lim_{x\to 0} \frac{1}{\sqrt{2+x}+\sqrt{2}} = \boxed{\frac{1}{2\sqrt{2}}}$
3. $\lim_{h\to 0} \frac{(x+h)^2-x^2}{h} = \lim_{h\to 0} \frac{x^2+2xh+h^2-x^2}{h} = \lim_{h\to 0} \frac{2xh+h^2}{h} = \lim_{h\to 0} \frac{h(2x+h)}{h} = \lim_{h\to 0}(2x+h) = \boxed{2x}$

Squeeze Theorem

If $g(x) \le f(x) \le h(x)$, that is, a function is bounded above and below by two other functions, and $\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$, that is, the two upper and lower functions have the same limit, then $\lim_{x\to c} f(x) = L$, that is, the limit of the function that is "squeezed" between the two functions with equal limits must have the same limit.

Example: Prove that $\lim_{x\to 0} x^2 \sin x = 0$ using the Squeeze Theorem.
Solution: $-1 \le \sin x \le 1$. Multiplying both sides of each inequality by x^2 yields $-x^2 \le x^2 \sin x \le x^2$.
Now, applying limits, we have $\lim_{x\to 0} -x^2 \ge \lim_{x\to 0} x^2 \sin x \ge \lim_{x\to 0} x^2 \;\leftrightarrow\; 0 \ge \lim_{x\to 0} x^2 \sin x \ge 0$
Since our limit is bounded above and below by 0, by the Squeeze theorem, we have that $\lim_{x\to 0} x^2 \sin x = 0$.

Two important limits that can be proven using the Squeeze Theorem that we will use to find many other limits are:

$$\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1} \quad and \quad \boxed{\lim_{x\to 0} \frac{1-\cos x}{x} = 0}$$

Examples:

1. $\lim_{x\to 0} \frac{\sin 2x}{\sin 3x} = \lim_{x\to 0} \frac{\sin 2x}{2x} \cdot \frac{3x}{\sin 3x} \cdot \frac{2}{3} = 1 \cdot 1 \cdot \frac{2}{3} = \boxed{\frac{2}{3}}$
2. $\lim_{h\to 0} \frac{\sin(x+h)-\sin x}{h} = \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} = \lim_{h\to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h} =$
   $\left[\lim_{h\to 0}(-\sin x) \cdot \frac{1-\cos h}{h}\right] + \left[\lim_{h\to 0}(\cos x) \cdot \frac{\sin h}{h}\right] = (-\sin x) \cdot 0 + \cos x \cdot 1 = \boxed{\cos x}$

Continuity

If $\lim_{x\to c} f(x) = f(c)$, we say that the function f is continuous at c.

Discontinuities occur when either
1. $f(x)$ is undefined,
2. $\lim_{x\to c} f(x)$ does not exist, or
3. $\lim_{x\to c} f(x) \neq f(c)$

If a discontinuity can be removed by inserting a single point, it is called removable. Otherwise, it is nonremovable (e.g. verticals asymptotes and jump discontinuities)

One-Sided Limits

$\lim_{x \to c^+} f(x) = L \quad limit\ from\ the\ right$

$\lim_{x \to c^-} f(x) = L \quad limit\ from\ the\ left$

Examples:

1. $\lim_{x \to 2^+} \frac{|x-2|}{x-2} = \boxed{1}$ Note that $\frac{|x-2|}{x-2} = \begin{cases} 1, & x \geq 2 \\ -1, & x < 2 \end{cases}$

2. $\lim_{x \to 1^+} f(x), where\ f(x) = \begin{cases} x, & x \leq 1 \\ 1-x, & x > 1 \end{cases}$ Since we want the limit from the right, we look at the piece where the function is defined > 1. Plugging 1 into $1 - x$ gives us that the limit is $\boxed{0}$.

Continuity at a point

A function f is continuous at c if the following 3 conditions are met:

1. $f(c)$ is defined
2. Limit of $f(x)$ exists when x approaches c
3. Limit of $f(x)$ when x approaches c is equal to $f(c)$

Continuity on an open interval

A function is continuous on an open interval if it is continuous at each point in the interval. A function that is continuous on the entire real line $(-\infty, \infty)$is everywhere continuous.

Continuity on a closed interval

A function f is continuous on the closed interval $[a, b]$ if it is continuous on the open interval l (a, b) and $\lim_{x \to a^+} f(x) = f(a)$ and $\lim_{x \to b^-} f(x) = f(b)$.

Examples:

1. Discuss the continuity of the function $f(x) = \frac{1}{x^2-4}$ on the closed interval $[-1,2]$. We know that this function has vertical asymptotes at $x = -2$ and $x = 2$, has a horizontal asymptote at $x = 0$, and has no zeros. The only discontinuities are at 2 and -2, and both are non-removable. So while we can't say that the function is continuous on $[-1,2]$ (because we would need the limit from the left to exist at $x = 2$), we can say that the function is continuous on $[-1,2)$.

2. Discuss the continuity of the function $f(x) = \frac{x-1}{x^2+x-2}$. Factoring yields $(x) = \frac{x-1}{(x-1)(x+2)}$. Since the $x - 1$ factors cancel, this tells us that there is a hole in the graph at $x = 1$, and that the function behaves like $f(x) = \frac{1}{x+2}$ everywhere except at $x = 1$. This latter function has a vertical asymptote at $x = -2$. Thus, our original function has a removable discontinuity at $x = 1$ and a non-removable discontinuity at $x = -2$.

Continuity of a Composite Function

If g is continuous at c and f is continuous at $g(c)$, then $(f \circ g)(x) = f\big(g(x)\big)$ is continuous at c.

Intermediate Value Theorem

If f is continuous on the closed interval $[a, b]$ and k is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a, b]$ such that $f(c) = k$.

Infinite Limits

$\lim_{x \to c} f(x) = \pm\infty$ means the function increases or decreases without bound; i.e. the graph of the function approaches a vertical asymptote

Finding Vertical Asymptotes

x-values at which a function is undefined result in either holes in the graph or vertical asymptotes. Holes result when a function can be rewritten so that the factor which yields the discontinuity cancels. Factors that can't cancel yield vertical asymptotes.

Example:

$f(x) = \frac{1}{x(x+3)}$ has vertical asymptotes at $x = 0$ and $x = 3$

$f(x) = \frac{(x+2)(x+3)}{x(x+3)}$ has a vertical asymptote at $x = 0$ and a hole at $x = -3$

Rules involving infinite limits

Let $\lim_{x \to c} f(x) = \infty$ and $\lim_{x \to c} g(x) = L$

1. $\lim_{x \to c}[f(x) \pm g(x)] = \infty$
2. $\lim_{x \to c}[f(x)g(x)] = \begin{cases} \infty, & L > 0 \\ -\infty, & L < 0 \end{cases}$
3. $\lim_{x \to c} \frac{g(x)}{f(x)} = 0$

The Derivative

The slope of the tangent line to the graph of f at the point (c, fc) is given by

$$m = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x}$$

The derivative of f at x is given by

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

An alternate form of the derivative that we often use to show that certain continuous functions are not differentiable at all points (either having sharp points or vertical tangent lines) is given by

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

Basic Differentiation Rules

1. The derivative of a constant function is zero, i.e., for $c \in \mathbb{R}$, $\frac{d}{dx}[c] = 0$
2. Power Rule for $n \epsilon \mathbb{Q}$, $\frac{d}{dx}[x^n] = nx^{n-1}$

   Special case: $\frac{d}{dx}[x] = 1$
3. Constant Multiple Rule $\in \mathbb{R}$, $\frac{d}{dx}[cf(x)] = cf'(x)$
4. Sum & Difference Rules $\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$

Derivatives of Trig Functions

1. $\frac{d}{dx}[\sin x] = \cos x$
2. $\frac{d}{dx}[\cos x] = -\sin x$
3. $\frac{d}{dx}[\tan x] = \sec^2 x$
4. $\frac{d}{dx}[\cot x] = -\csc^2 x$
5. $\frac{d}{dx}[\sec x] = \sec x \tan x$
6. $\frac{d}{dx}[\csc x] = -\csc x \cot x$

Rates of Change

The average rate of change of a function f with respect to a variable t is given by $\frac{\Delta f}{\Delta t} = \frac{f(t_2)-f(t_1)}{t_2-t_1}$

The instantaneous rate of change of a function f at an instance of the variable t is given by $f'(t)$

The rate of change of position is velocity.
The rate of change of velocity is acceleration.
The position of a free-falling object under the influence of gravity is given by $s(t) = \frac{1}{2}gt^2 + v_0 t + s_o$, where $s_o =$ initial height of the object, $v_0 =$ initial velocity of the object, $g = -32\frac{ft}{s^2} or - 9.8\ m/s^2$

Product Rule

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + f'(x)g(x)$$

Quotient Rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}$$

"low dee high less high dee low, draw the line and square below"

Chain Rule

$$\frac{d}{dx}[f(g(x))] = f'\big(g(x)\big)g'(x)$$

When one or more functions are composed, you take the derivative of the outermost function first, keeping its inner function, and then multiply that entire thing by the derivative of the inner function. If more than two functions are composed, you just repeat this process for all inner functions, e.g.

$$\frac{d}{dx}[f(g(h(x)))] = f'\Big(g\big(h(x)\big)\Big)\,g'\big(h(x)\big)h'(x)$$

$$\frac{d}{dx}[f(g(h(k(x))))] = f'\bigg(g\Big(h\big(k(x)\big)\Big)\bigg)\,g'\Big(h\big(k(x)\big)\Big)\,h'\big(k(x)\big)k'(x)$$

Example:
$f(x) = \sin\left[\sqrt{3\sec^2(5x^4 - 7x) + \cos x}\right]$

First we want to rewrite any roots as fractional powers, and any $f^n(x)$ as $[f(x)]^n$.
$f(x) = \sin\left[(3\,[\sec(5x^4 - 7x)]^2 + \cos x)^{1/2}\right]$

Let's identify our sequence of nested functions to determine what order we will take derivatives.
$\sin a \quad , \quad a = b^{1/2} \quad , \quad b = 3c^2 + \cos x \quad , \quad c = \sec d \quad , \quad d = 5x^4 - 7x$

The derivatives of these functions are, respectively,
$(\cos a)a' \;,\; a' = \left(\frac{1}{2}b^{-1/2}\right)b' \quad , \quad b' = (6c)c' - \sin x \quad , \quad c' = (\sec d \tan d)d' \quad , \quad d' = 20x^3 - 7$

So, using the chain rule, the derivative of our original function is

$$f'(x) = \left(\cos\left[\sqrt{3\sec^2(5x^4 - 7x) + \cos x}\right]\right) \cdot \left(\frac{1}{2}(3\,[\sec(5x^4 - 7x)]^2 + \cos x)^{-1/2}\right) \cdot$$
$$\cdot ([6\sec(5x^4 - 7x)] \cdot (\sec(5x^4 - 7x)\tan(5x^4 - 7x)) \cdot (20x^3 - 7) - \sin x)$$

Implicit Differentiation

When taking the derivative of a function with respect to x, the only variable whose derivative is 1 is x, since all other variables are treated as functions of x (except variables standing for constants, like π). So, when we have an equation in x and y written implicitly (i.e. not solved for y in terms of x) and we want to find y', we take the derivative of both sides, but every derivative involving y must treat y as a function of x, meaning it has to be multiplied by the derivative of the "inside" function, y'.

Example:
$x^2 + y^3 = 3xy$

Taking the derivative of both sides with respect to x, we have
$2x + 3y^2y' = 3xy' + 3y$ (we used the power rule twice on the LHS, and the product rule on the RHS)
Then we put all our y'-terms on one side, factor out the y', and divide.
$3y^2y' - 3xy' = 3y - 2x$
$y'(3y^2 - 3x) = 3y - 2x$

$$\boxed{y' = \frac{3y - 2x}{3y^2 - 3x}}$$

If we want to find y'' in terms of x and y, we just take the derivative of y', again remembering to treat y as a function of x. We'll end up with an expression in x, y, and y', so we substitute the expression we found for y' in terms of just x and y. Using the above implicit differentiation for y', we have

$$y'' = \frac{(3y^2 - 3x)(3y' - 2) - (3y - 2x)(6yy' - 3)}{(3y^2 - 3x)^2}$$
$$= \frac{(3y^2 - 3x)\left(3\left[\frac{3y - 2x}{3y^2 - 3x}\right] - 2\right)(3y - 2x)\left(6y\left[\frac{3y - 2x}{3y^2 - 3x}\right] - 3\right)}{(3y^2 - 3x)^2}$$

Related Rates

Related rate problems are basically just applied implicit differentiation problems, typically dealing with 2 or more variables that are each functions of an additional variable. For example, a problem dealing with the volume of a cone, where volume, radius, and height are all functions of time. We take the derivative of both sides of our equation implicitly, remembering that each variable is a function of time.

Basic equations/formulas to know:

Volume of a Sphere	$V = \frac{4}{3}\pi r^3$
Surface Area of a Sphere	$A = 4\pi r^2$
Volume of a cone	$V = \frac{1}{3}\pi r^2 h$
Volume of a right prism	$V = (area\ of\ base) \cdot (perpendicular\ height)$

Example problem: A conical tank is 10 feet across at the top and 10 feet deep. If it is being filled with water at a rate of 5 cubic feet per minute, find the rate of change of the depth of the water when it is 3 feet deep.

Steps to solve:

1. Identify knowns/unknowns

$$r = 5\ when\ h = 10 \ \rightarrow\ \frac{r}{h} = \frac{5}{10} = \frac{1}{2} \ \rightarrow\ r = \frac{h}{2}$$

$$\frac{dv}{dt} = 5\ ft^3/min$$

$$\frac{dh}{dt} = ?\ \ when\ h = 3ft$$

2. Identify equation

$$V = \frac{1}{3}\pi r^2 h$$

Note that this equation involves variables V, r, and h, all of which are functions of t. When we take the derivative, we will end up with $\frac{dv}{dt}, \frac{dh}{dt}, and\ \frac{dr}{dt}$. We know the value of $\frac{dv}{dt}$, and we're trying to solve for $\frac{dh}{dt}$, but we don't know anything about $\frac{dr}{dt}$, so we want to rewrite the equation so that it doesn't include r at all. To do this, we use the ratio of radius to height of the tank. In other problems, this step may include rearranging the Pythagorean theorem, or using trig functions - whatever information you know about the given shape/area/volume.

3. Rewrite equation

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{1}{3}\left(\frac{1}{4}\right)\pi h^3$$

4. Take the derivative of both sides with respect to t (or whatever the independent variable is).

$$\frac{dV}{dt} = \frac{1}{4}\pi h^2 \cdot \frac{dh}{dt}$$

5. Rearrange to solve for the unknown.

$$\frac{dh}{dt} = \frac{4\frac{dV}{dt}}{\pi h^2}$$

6. Plug in known values. Note that we are not plugging in any known values until the VERY end of the problem. Otherwise, our derivative would have been inaccurate.

$$\frac{dh}{dt} = \frac{4(5)}{\pi 3^2} = \frac{20}{9\pi}\ ft/min$$

<u>Relative Extrema, Increasing & Decreasing, and the First Derivative Test</u>

Recall that the derivative of a function at a point is the slope of the tangent line at that point. When the derivative is zero, the slope is zero, when the derivative is positive, the slope is positive, and when the derivative is negative, the slope is negative.

$f'(x)$	$f(x)$
$f'(x) = 0$	f is either constant or has a relative maximum or minumum
$f'(x) > 0$	f is increasing
$f'(x) < 0$	f is decreasing

Absolute Extrema on a Closed Interval

To find absolute maxima and minima on a closed interval (the highest and lowest y-values a function achieves on the interval), first take the derivative of the function, set it equal to zero, and solve for x. These values are called <u>critical values</u> or critical numbers. Then plug any of these x-values which lie within the given closed interval, and the x-values of the endpoints of the interval into the original function in order to find the y-values of these points. The largest y-value is your <u>absolute maximum</u> and the smallest y-value is your <u>absolute minimum</u>.

Open Intervals on which a Function is Increasing or Decreasing

If we want to discuss the behavior of a function on its entire domain, we take all of the critical numbers we found by setting the derivative equal to zero, along with any other x-values for which the derivative is undefined, and use these numbers to split the number line into intervals. For example, say we have a function with the following derivative:

$$f'(x) = \frac{(x-2)(x+5)(x-8)}{(x+1)^2}$$

The critical numbers are $-5, -1, 2,$ and 8. So, we split up the real number line into the following intervals, and then choose a number within each interval to plug into the derivative to see if it is positive or negative in that interval (if it is positive/negative for one value in that interval, it will be for all values). Note that the denominator is always positive, so we can concentrate on the numerator to determine if f' is positive/negative for each value.

$(-\infty, -5)$	$(-5, -1)$	$(-1,2)$	$(2,8)$	$(8, \infty)$
$f'(-6) < 0$	$f'(-2) > 0$	$f'(0) > 0$	$f'(3) < 0$	$f'(9) > 0$

Thus, the function f is increasing on $(-5, -1) \cup (-1,2) \cup (8, \infty)$ and decreasing on $(-\infty, -5) \cup (2,8)$.

Concavity and the Second Derivative

The second derivative is the rate of change of the slope of a function. When slope increases, a function is concave up. When slope decreases, the function is concave down. When a function changes concavity, we get what are called inflection points. To determine possible inflection points, take the second derivative of the function, set it equal to zero and solve for x. Those values are the x-coordinates of your inflection points. Then, divide up the real number line into intervals based on those numbers, and plug a number from each interval into the second derivative. If positive, the function is concave up on that interval. If negative, the function is concave down on the interval.

$f''(x)$	$f(x)$
$f''(x) = 0$	f has inflection point
$f''(x) > 0$	f is concave up
$f''(x) < 0$	f is concave down

The second derivative can also be used to determine whether a particular relative extremum, found using the first derivative, is in fact a maximum or a minimum, without having to determine whether the function is increasing or decreasing on either side of the extremum. If the function is concave down at an extremum, it is a maximum; if the function is concave up at an extremum, it is a minimum.

Optimization Problems

When trying to maximize or minimize a function, you're basically using the first derivative test to find relative extrema. When you set the first derivative equal to zero and solve for x, if there is more than one solution, it will usually be obvious that only one of them makes sense as the answer. For example, a negative value for area, radius, etc. certainly doesn't make sense, so you choose the positive value.

These problems will typically involve formulas with multiple variables, and so require multiple equations to solve. Before taking any derivatives, figure out every equation you can that relate the variables, then rewrite the formula you are trying to maximize or minimize by substituting for one or more of the variables so that you end up with a function of a single variable.

Rolle's Theorem: Let f be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). If $f(a) = f(b)$, then there is at least one number c in (a, b) such that $f'(c) = 0$.

Basically, Rolle's Theorem states that if a continuous, differentiable function achieves the same y-value at both endpoints of a given interval, then there must be some value in that interval there the function has a horizontal tangent line.

Mean Value Theorem: If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists a number c in (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

The Mean Value Theorem is really a generalized version of Rolle's Theorem. It states that if a function is continuous and differentiable on a given interval, then there must be some value in that interval where the slope of the tangent line to the graph of the function is the same as the slope of the secant line through the enpoints of the interval.

Limits at Infinity

The line $y = L$ is a horizontal asymptote of the graph of f if $\lim_{x\to-\infty} f(x) = L$ or $\lim_{x\to\infty} f(x) = L$.
If r is a positive rational number and c is any real number, then $\lim_{x\to\infty} \frac{c}{x^r} = 0$. If x^r is defined when $x < 0$, then $\lim_{x\to-\infty} \frac{c}{x^r} = 0$.

Guidelines for finding limits at infinity of rational functions:

If the degree of the numerator is less than the degree of the denominator, then the limit is 0.
If the degree of the numerator is equal to the degree of the denominator, then the limit is the ratio of leading coefficients.
If the degree of the numerator is larger than the degree of the denominator, then the limit is $\pm\infty$.
(the above 3 rules follow the rules for horizontal/oblique asymptotes we learned in Precal)

If the numerator or denominator has a radical, remember that the nth root of x to the n is defined as:

$$\sqrt[n]{x^n} = \begin{cases} x, & if\ n\ is\ odd \\ |x|, & if\ n\ is\ even \end{cases}$$

and that absolute value of x is defined as:

$$|x| = \begin{cases} x, & if\ x \geq 0 \\ -x, & if\ x < 0 \end{cases}$$

Indeterminate Forms and l'Hopital's Rule

$0/0$, ∞/∞, $0 \cdot \infty$, 1^∞, 0^0, and $\infty - \infty$ are called indeterminate forms.

l'Hopital's Rule: Let f and g be functions that are differentiable on an open interval (a, b) containing c, except possibly at c itself. Assume that $g'(x) \neq 0$ for all x in (a,b), except possibly at c itself. If the limit of f(x)/g(x) as x approaches c produces an indeterminate form 0/0, ∞/∞, $(-\infty)/\infty$, or $\infty/(-\infty)$, then
$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$

DERIVATIVE RULES

Power Rule:

$$\frac{d}{dx}[x^n] = nx^{n-1}$$

Constant Multiple Rule:

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)]$$

Sum & Difference:

$$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$$

Product Rule:

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$$

Quotient Rule:

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$$

Chain Rule:

$$\frac{d}{dx}[f(g(x))] = f'\big(g(x)\big)g'(x)$$

Trig Functions:

$$\frac{d}{dx}[\sin x] = \cos x \qquad \frac{d}{dx}[\tan x] = \sec^2 x \qquad \frac{d}{dx}[\sec x] = \sec x \tan x$$

$$\frac{d}{dx}[\cos x] = -\sin x \qquad \frac{d}{dx}[\cot x] = -\csc^2 x \qquad \frac{d}{dx}[\csc x] = -\csc x \cot x$$

Exponential and Logarithmic Functions:

$$\frac{d}{dx}[a^u] = a^u u' \ln a \qquad \frac{d}{dx}[\log_a u] = \frac{u'}{u \ln a}$$

Inverse Functions:

$$\frac{d}{dx}[f^{-1}(x)] = \frac{1}{f'(f^{-1}(x))}$$

Inverse Trig Functions:

$$\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1-u^2}} \qquad \frac{d}{dx}[\arccos u] = \frac{-u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx}[\arctan u] = \frac{u'}{1+u^2} \qquad \frac{d}{dx}[\operatorname{arccot} u] = \frac{-u'}{1+u^2}$$

$$\frac{d}{dx}[\operatorname{arcsec} u] = \frac{u'}{|u|\sqrt{u^2-1}} \qquad \frac{d}{dx}[\operatorname{arccsc} u] = \frac{-u'}{|u|\sqrt{u^2-1}}$$