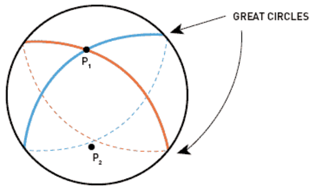
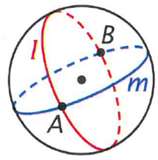
Ch 16 - Non-Euclidean Geometries



16.1 - Spherical or Riemannian Geometry

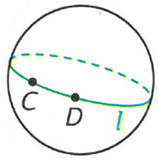
Def: A great circle of a sphere is a set of points that is the intersection of the sphere and a plane containing its center.

Def: Antipodal points are the two points of intersection of a sphere with a line through its center.



A postulate of Euclidean Geometry states that "two points define a line."

If we think about the surface of a sphere as a "plane" and great circles as "lines," we can see that this postulate does not hold on the sphere, as through any two antipodal points, we can draw infinitely many great circle lines.



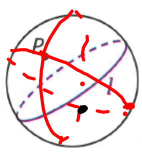
If we think about a pair of antipodal points as a single point, then the postulate is true again.

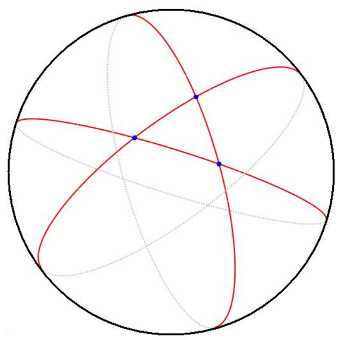
Euclid's Parallel Postulate also fails in spherical geometry.

Again, thinking about a pair of antipodal points as a single point and great circles as lines,

if we look at a point P not on a line ℓ, there are no lines that can be drawn through P that are parallel to ℓ-- every great circle of a sphere intersects all other great circles on that sphere!

Spherical geometry is considered a non-Euclidean geometry because these postulates that hold in Euclidean geometry fail.

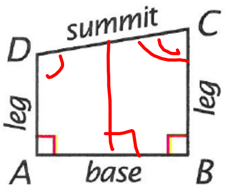


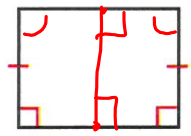


16.2 - The Saccheri Quadrilateral

Def: A birectangular quadrilateral is a quadrilateral that has two sides perpendicular to a third side.

Def: A Saccheri quadrilateral is a bicrectangular quadrilateral whose legs are equal.





A Saccheri quadrilateral

Theorem 87: The summit angles of a Saccheri quadrilateral are equal.

Theorem 88: The line segment connecting the midpoints of the base and summit of a Saccheri quadrilateral is perpendicular to both of them.

Theorem 89: If the legs of a birectangular quadrilateral are unequal, the summit angles opposite them are unequal in the same order.

Theorem 90: If the summit angles of a birectangular quadrilateral are unequal, the legs opposite them are unequal in the same order.

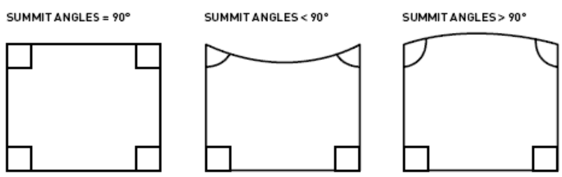
16.3 - Hyperbolic or Lobachevskian Geometry

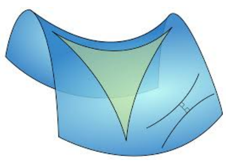
The Lobachevskian Postulate: The summit angles of a Saccheri quadrilateral are acute.

Lobachevskian Theorem 1: The summit of a Saccheri quadrilateral is longer than its base.

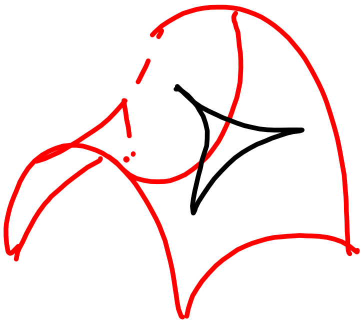
Lobachevskian Theorem 2: A midsegment of a triangle is less than half as long as the third side.

Statement	**Euclid**	**Lobachevsky**	**Riemann**
Through a point not on a line, there is	*exactly one* line parallel to the line.	*more than one* line parallel to the line.	*no line* parallel to the line.
The summit angles of a Saccheri quadrilateral are	right.	acute.	obtuse.



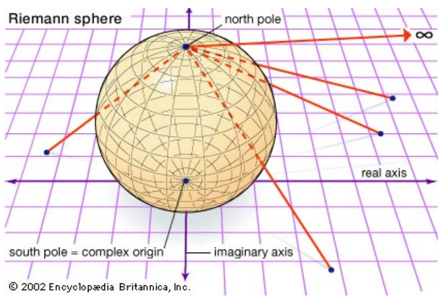


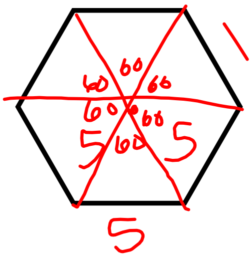
Σ angles in a triange
$> 180°$

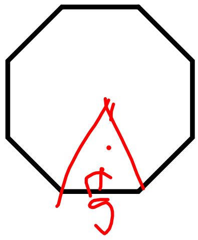


Σ angles in a △
$< 180°$

Circle Limit III, 1958 M.C. Escher, Cordon Art (c) 2002







$$\frac{\text{linear measurement of a}}{\text{linear measurement of b}} = \frac{r_1}{r_2} = \frac{l_1}{l_2}$$

$$\frac{A_1}{A_2} = \frac{(l_1)^2}{(l_2)^2} \; ; \; \frac{V_1}{V_2} = \frac{(l_1)^3}{(l_2)^3}$$

A_1 A_2

V_1 V_2