6.1 Identities: Pythagorean & Sum and Difference

An identity is an equation that is true for all elements in its domain.

Some identities that we have already seen:

Reciprocal Identities

$\csc x = \frac{1}{\sin x}$, $\sin x = \frac{1}{\csc x}$, $\sec x = \frac{1}{\cos x}$, $\cos x = \frac{1}{\sec x}$, $\cot x = \frac{1}{\tan x}$, $\tan x = \frac{1}{\cot x}$

Ratio Identities

$\tan x = \frac{\sin x}{\cos x}$, $\cot x = \frac{\cos x}{\sin x}$

Cofunction Identities

$\sin\left(\frac{\pi}{2} - x\right) = \cos x$, $\cos\left(\frac{\pi}{2} - x\right) = \sin x$

$\tan\left(\frac{\pi}{2} - x\right) = \cot x$, $\cot\left(\frac{\pi}{2} - x\right) = \tan x$

$\csc\left(\frac{\pi}{2} - x\right) = \sec x$, $\sec\left(\frac{\pi}{2} - x\right) = \csc x$

$\frac{\pi}{2} - x = 90^\circ - x$ = complement of x

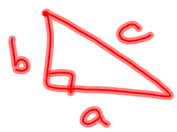
"function of an angle = cofunction of its complement"

Pythagorean Identities

What does the word "Pythagorean" make you think of?

Pythagorean Theorem

$a^2 + b^2 = c^2$



Recall that the coordinates (x,y) of a point on the unit circle correspond exactly to (cosΘ, sinΘ) for the angle Θ whose terminal side passes through that point.

$(x,y) = (\cos\theta, \sin\theta)$

$(\sin\theta)^2 + (\cos\theta)^2 = 1^2$

$$\boxed{\sin^2\theta + \cos^2\theta = 1}$$

$[f(x)]^2 = f^2(x) \neq f(x)^2 = f(x^2)$

$$\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$$

The other two Pythagorean Identities are derived from the first.

$$\frac{\sin^2 x + \cos^2 x}{\sin^2 x} = \frac{1}{\sin^2 x}$$

$$\frac{\sin^2 x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x} = \frac{1}{\sin^2 x}$$

$$\boxed{1 + \cot^2 x = \csc^2 x}$$

$$\frac{\sin^2 x + \cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$\boxed{\tan^2 x + 1 = \sec^2 x}$$

Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1 \quad , \quad 1 + \cot^2 x = \csc^2 x \quad , \quad \tan^2 x + 1 = \sec^2 x$$

Proving Identities

To prove an identity means to show that one side of the identity can be rewritten in a form that is identical to the other side.

There is no one method that works for every identity, the following are some helpful guidelines:

- remember that you are trying to prove that this equation is true, so you can't treat it like an equation -- no working on both sides (e.g. you can't add something to both sides, or divide both sides by something). You must start with one side and rewrite it until it is equal to the other side. It is okay to meet in the middle if you get stuck and must work from both sides.
- if one side is more complicated than the other, start with the more complicated side and try to simplify it
- use rules of algebra to find common denominators, add fractions, square binomials, factor, multiply by a form of 1, add 0, etc.
- apply known identities to rewrite parts of an expression in a more useful form, e.g. since $\sin^2 x + \cos^2 x = 1$, you can replace the expression $\sin^2 x + \cos^2 x$ with 1.
- when in doubt, rewrite in terms of sine and cosine

Pythagorean Identities

$\sin^2 x + \cos^2 x = 1$, $1 + \cot^2 x = \csc^2 x$, $\tan^2 x + 1 = \sec^2 x$

Useful formulas from Algebra:

$a^2 - b^2 = (a-b)(a+b)$

$(a+b)^2 = a^2 + 2ab + b^2$

$(a-b)^2 = a^2 - 2ab + b^2$

$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

$(a+b)^2 = (a+b)(a+b)$
$\neq a^2 + b^2$

6.1 Handout Prove.

14. $\sin x \cot x \sec x = 1$

$LHS = \sin x \cot x \sec x = \frac{\cancel{\sin x}}{1} \cdot \frac{\cancel{\cos x}}{\cancel{\sin x}} \cdot \frac{1}{\cancel{\cos x}} = 1 = RHS$ ✓

22. $\frac{\sin x}{1-\cos x} = \csc x + \cot x$

$LHS = \frac{\sin x}{(1-\cos x)} \cdot \frac{(1+\cos x)}{(1+\cos x)}$

$= \frac{\sin x(1+\cos x)}{1-\cos^2 x}$

$= \frac{\cancel{\sin x}(1+\cos x)}{\sin^{\cancel{2}} x}$

$= \frac{1+\cos x}{\sin x} = \frac{1}{\sin x} + \frac{\cos x}{\sin x}$

$= \csc x + \cot x = RHS$ ✓

$\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$

$\cancel{\frac{a}{b+c} \neq \frac{a}{b} + \frac{a}{c}}$

$\sin^2 x + \cos^2 x = 1$
$\sin^2 x = 1 - \cos^2 x$
$\cos^2 x = 1 - \sin^2 x$

$(a+b)(a-b)$
$= a^2 - b^2$

$\frac{\cancel{x}}{\cancel{x^2}} = \frac{1}{x}$

42. $\dfrac{1}{\tan^2 x} - \dfrac{1}{\cot^2 x} = \csc^2 x - \sec^2 x$

$$\text{LHS} = \frac{1}{\left(\frac{\sin^2 x}{\cos^2 x}\right)} - \frac{1}{\left(\frac{\cos^2 x}{\sin^2 x}\right)} =$$

$$= \frac{\cos^2 x}{\sin^2 x} - \frac{\sin^2 x}{\cos^2 x}$$

$$= \frac{\cos^2 x}{\sin^2 x} \cdot \frac{\cos^2 x}{\cos^2 x} - \frac{\sin^2 x}{\cos^2 x} \cdot \frac{\sin^2 x}{\sin^2 x}$$

$$\text{LHS} = \cot^2 x - \tan^2 x$$

$$= (\csc^2 x - 1) - (\sec^2 x - 1)$$

$$= \csc^2 x - \sec^2 x$$

$$= \text{RHS} \checkmark$$

$\tan^2 x + 1 = \sec^2 x$
$\tan^2 x = \sec^2 x - 1$
$\cot^2 x + 1 = \csc^2 x$
$\cot^2 x = \csc^2 x - 1$

66. $\dfrac{\sin^3 x - \cos^3 x}{\sin x + \cos x} = \dfrac{\csc^2 x - \cot x - 2\cos^2 x}{1 - \cot^2 x}$

$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

$$\text{LHS} = \frac{(\sin x - \cos x)(\sin^2 x + \sin x \cos x + \cos^2 x)}{\sin x + \cos x}$$

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$$= \frac{(\sin x - \cos x)(1 + \sin x \cos x)}{\sin x + \cos x}$$

$\dfrac{ab}{c} = \dfrac{a}{c} \cdot b$

$a \cdot \dfrac{b}{c}$

$$= \frac{\sin x + \sin^2 x \cos x - \cos x + \sin x \cos^2 x}{\sin x + \cos x}$$

$$= \frac{\sin x + (1 - \cos^2 x)\cos x - \cos x + \sin x(1 - \sin^2 x)}{\sin x + \cos x}$$

$$= \frac{\sin x + \cancel{\cos x} - \cos^3 x - \cancel{\cos x} + \sin x - \sin^3 x}{\sin x + \cos x}$$

Homework:

6.1 Handout #1-70, at LEAST all the odds, but the more of these you work, the easier they will become.

Note that for most of the sections in the handout, answers to all problems (even and odd) are given, but there are no solutions given to any of the proof problems. Please see section 6.3 in your textbook for additional worked examples and solutions to odd-numbered proof exercises.

Reminders:

Word Problem Workshop TODAY at 3:45 in S201