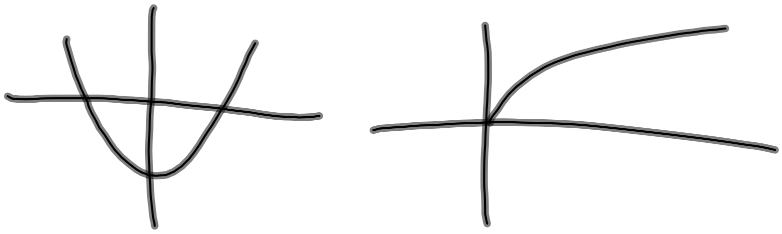
Inverse Trigonometric Functions

(6.4 book / 6.5 handout)

Recall from Algebra:

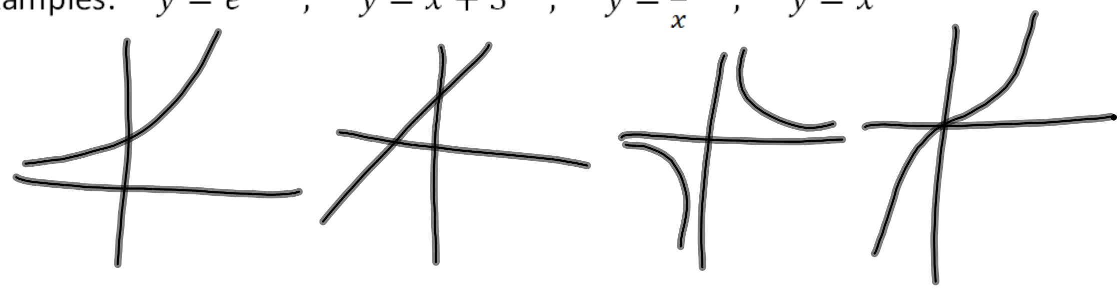
- f is a **function** if each input value (x) has a unique output $f(x)$.

  Examples: $f(x) = x^2 - 2$, $f(x) = \sqrt{x}$



- f is **one-to-one** if, in addition, each y corresponds to only one x.

  Examples: $y = e^x$, $y = x + 3$, $y = \frac{1}{x}$, $y = x^3$



- If f is a one-to-one function, we can define its inverse $f^{-1}(x)$.

  Note that this notation is not exponentiation, i.e. $f^{-1}(x) \neq \frac{1}{f(x)}$
- $f(x)$ and $g(x)$ are **inverses** if

  $(f \circ g)(x) = f\big(g(x)\big) = x = g\big(f(x)\big) = (g \circ f)(x)$,

  that is, **inverse functions "undo" each other**.

Example: $f(x) = x^3$, $g(x) = \sqrt[3]{x}$

$$(f \circ g)(x) = f(g(x)) = \left[\sqrt[3]{x}\right]^3 = x$$

$$(g \circ f)(x) = g(f(x)) = \sqrt[3]{x^3} = x$$

What do we mean by an Inverse Trig function?

Recall that **for a basic Trigonometric function**, e.g. $f(x) = \sin x$,

- **The input (x) is an angle**
- **The output $f(x)$ is a ratio of sides**

So **for an inverse Trigonometric function,**

- **The input (x) is a ratio of sides**
- **The output $f(x)$ is an angle**

Construction of the inverse of $f(x) = \sin x$:

$y = \sin x$

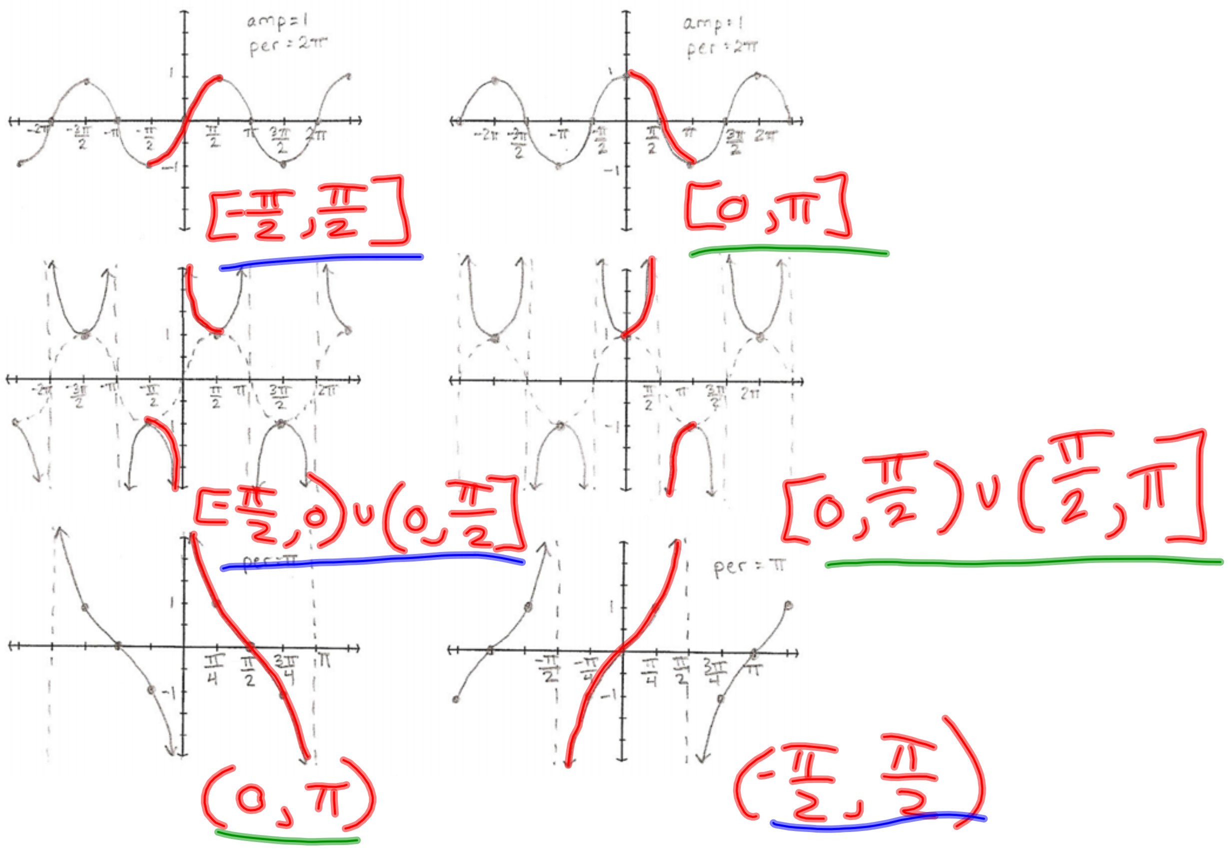
$x = \sin y$

$y =$ the angle whose sine value is x

$f^{-1}(x) = \sin^{-1}(x)$ or $f^{-1}(x) = \arcsin(x)$

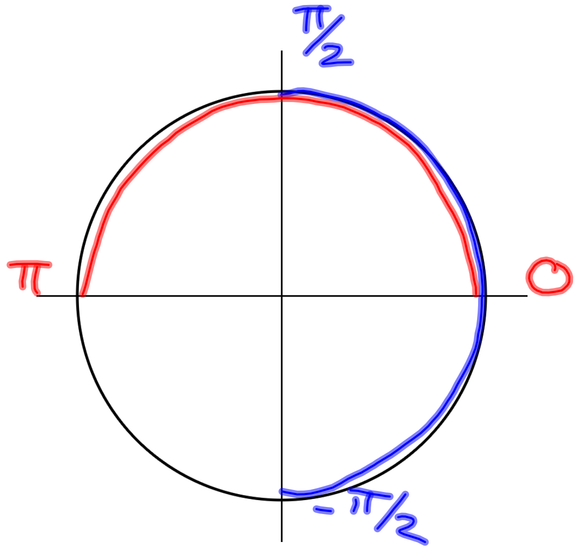
* $\sin^{-1}(x) \neq \dfrac{1}{\sin x}$

But Trigonometric functions aren't one-to-one – how is the inverse defined? We must restrict the domain!



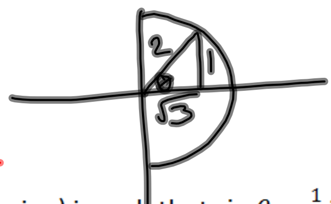
Summary of Restricted Domains:

Interval	Functions	Quadrants
$\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$	$\sin x, \csc x, \tan x$	IV & I
$(0,\pi)$	$\cos x, \sec x, \cot x$	I & II



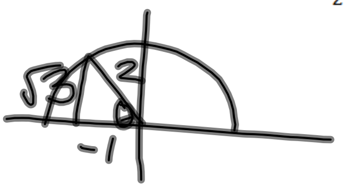
Evaluate the inverse trigonometric expression.

$\sin^{-1}\left(\frac{1}{2}\right) = 30° = \frac{\pi}{6} \Leftrightarrow \sin 30° = \frac{1}{2}$



In words: What angle θ, between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ (the restricted domain for sine) is such that $\sin\theta = \frac{1}{2}$?

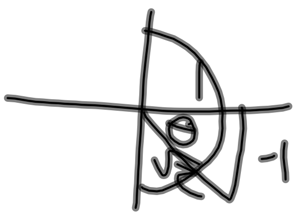
$\cos^{-1}\left(-\frac{1}{2}\right) = 120° = \frac{2\pi}{3}$



In words: What angle θ, between 0 and π (the restricted domain for cosine) is such that $\cos\theta = -\frac{1}{2}$?

$\tan^{-1}(-1) = \boxed{-\frac{\pi}{4}} = -45°$

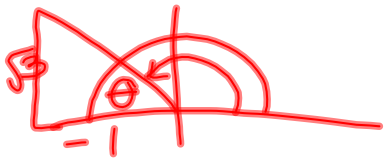
What $\angle\theta$ betw $-\frac{\pi}{2}$ & $\frac{\pi}{2}$ is s.t. $\tan\theta = -1$?



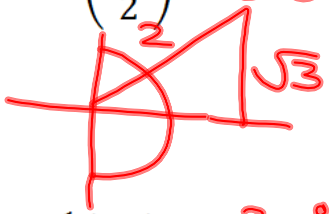
$315° / \frac{7\pi}{4}$ is not between $-\frac{\pi}{2}$ & $\frac{\pi}{2}$

Evaluate.

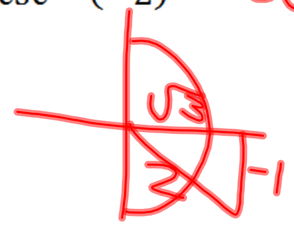
$\cot^{-1}\left(-\frac{1}{\sqrt{3}}\right) = 120° = \frac{2\pi}{3}$



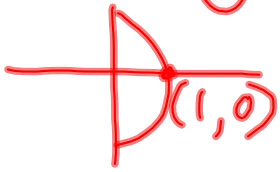
$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = 60° = \frac{\pi}{3}$



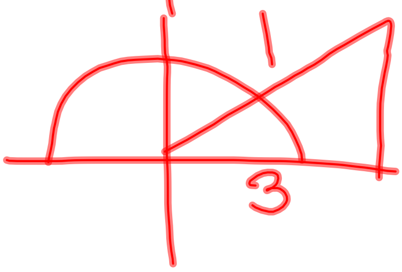
$\csc^{-1}(-2) = -30° = -\frac{\pi}{6}$

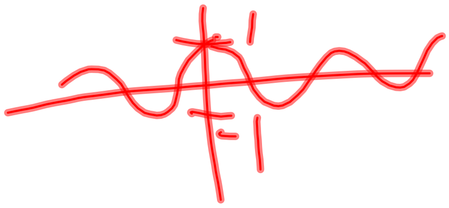


$\tan^{-1}(0) = 0° = 0$



$\cos^{-1}(3) =$ undefined





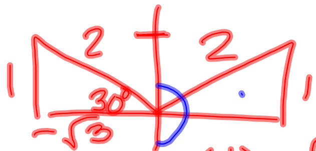
What happens when we compose a Trigonometric function with its inverse?

According to the definition,

$f(x)$ and $g(x)$ are inverses if $f(g(x)) = x$ and $g(f(x)) = x$
(for all x-values in the respective domains of g and f)

We would then expect

$\sin(\sin^{-1} x) = x$ and $\sin^{-1}(\sin x) = x$



$\sin\left(\sin^{-1}\frac{1}{2}\right) = \sin\frac{\pi}{6} = \boxed{\frac{1}{2}}$

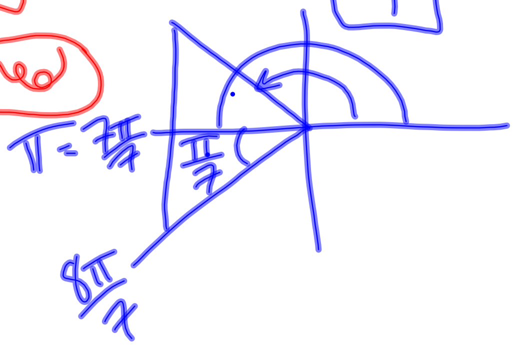
$\sin^{-1}\left(\sin\left(\frac{5\pi}{6}\right)\right) = \sin^{-1}\left(\frac{1}{2}\right) = \boxed{\frac{\pi}{6}}$

$\sin^{-1}\left(\sin\left(-\frac{\pi}{6}\right)\right) = \boxed{-\frac{\pi}{6}}$

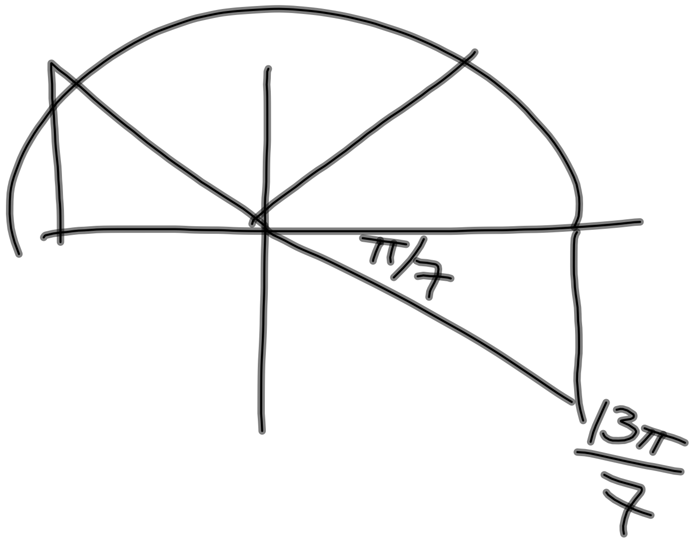
$-\frac{1}{2}, = \sin^{-1}\left(-\frac{1}{2}\right)$

$\cos^{-1}\left(\cos\left(\frac{8\pi}{7}\right)\right) = \boxed{\frac{6\pi}{7}}$

$\sin(\sin^{-1} 3) =$ undefined



$$\cot^{-1}\left(\cot\frac{13\pi}{7}\right) \stackrel{?}{=} \boxed{-\frac{6\pi}{7}}$$



Homework:

6.5 Handout #1-24

ALL!