Chapter 6 - Trigonometric Identities and Equations

Reciprocal Identities

$$\csc x = \frac{1}{\sin x} \quad , \quad \sin x = \frac{1}{\csc x}$$

$$\sec x = \frac{1}{\cos x} \quad , \quad \cos x = \frac{1}{\sec x}$$

$$\cot x = \frac{1}{\tan x} \quad , \quad \tan x = \frac{1}{\cot x}$$

Cofunction Identities

$$\sin\left(\frac{\pi}{2} - x\right) = \cos x \quad , \quad \cos\left(\frac{\pi}{2} - x\right) = \sin x$$

$$\tan\left(\frac{\pi}{2} - x\right) = \cot x \quad , \quad \cot\left(\frac{\pi}{2} - x\right) = \tan x$$

$$\csc\left(\frac{\pi}{2} - x\right) = \sec x \quad , \quad \sec\left(\frac{\pi}{2} - x\right) = \csc x$$

Ratio Identities

$$\tan x = \frac{\sin x}{\cos x} \quad , \quad \cot x = \frac{\cos x}{\sin x}$$

Odd-Even Identities

$$\cos(-x) = \cos x \quad , \quad \sin(-x) = -\sin x \quad , \quad \tan(-x) = -\tan x$$
$$\sec(-x) = \sec x \quad , \quad \csc(-x) = -\csc x \quad , \quad \cot(-x) = -\cot x$$

Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1$$
$$1 + \cot^2 x = \csc^2 x$$
$$\tan^2 x + 1 = \sec^2 x$$

Proving Identities

To prove an identity means to show that one side of the identity can be rewritten in a form that is identical to the other side.

There is no one method that works for every identity, the following are some helpful guidelines:

- remember that you are trying to prove that this equation is true, so you can't treat it like an equation -- no working on both sides (e.g. you can't add something to both sides, or divide both sides by something). You must start with one side and rewrite it until it is equal to the other side. It is okay to meet in the middle if you get stuck and must work from both sides.
- if one side is more complicated than the other, start with the more complicated side and try to simplify it
- use rules of algebra to find common denominators, add fractions, square binomials, factor, multiply by a form of 1, add 0, etc.
- apply known identities to rewrite parts of an expression in a more useful form, e.g. since $\sin^2 x + \cos^2 x = 1$, you can replace the expression $\sin^2 x + \cos^2 x$ with 1.
- when in doubt, rewrite in terms of sine and cosine

Useful formulas from Algebra:

$$a^2 - b^2 = (a - b)(a + b)$$

$$(a + b)^2 = a^2 + 2ab + b^2$$

$$(a - b)^2 = a^2 - 2ab + b^2$$

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

$$(a+b)^2 \neq a^2 + b^2$$

$$(a+b)(a+b)$$

6.1 Prove.

$$\sin^2 x + \cos^2 x = 1$$
$$\ast \sin^2 x = 1 - \cos^2 x$$

14. $\sin x \cot x \sec x = 1$

$$LHS = \frac{\sin x}{1} \cdot \frac{\cos x}{\sin x} \cdot \frac{1}{\cos x} = 1 = RHS \checkmark$$

22. $\dfrac{\sin x}{1 - \cos x} = \csc x + \cot x$

$$LHS = \frac{\sin x}{(1 - \cos x)} \cdot \frac{(1 + \cos x)}{(1 + \cos x)} = \frac{\sin x (1 + \cos x)}{1 - \cos^2 x}$$

$$= \frac{\sin x (1 + \cos x)}{\sin^2 x} = \frac{1 + \cos x}{\sin x} = \frac{1}{\sin x} + \frac{\cos x}{\sin x}$$

$$= \csc x + \cot x = RHS \checkmark$$

$$\frac{a + b}{c} = \frac{a}{c} + \frac{b}{c}$$

6.2 - Sum and Difference Identities

$$\sin(a+b) = \sin a \cos b + \cos a \sin b$$

$$\sin(a-b) = \sin a \cos b - \cos a \sin b$$

$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$

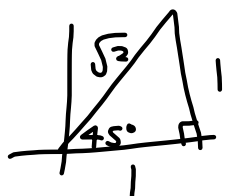$$\cos(a-b) = \cos a \cos b + \sin a \sin b$$

$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$$

$$\tan(a-b) = \frac{\tan a - \tan b}{1 + \tan a \tan b}$$

2. $\sin 375° = \sin(\overset{a}{330°} + \overset{b}{45°})$

$$\sin(a+b) = \sin a \cos b + \cos a \sin b$$

$$\sin(330° + 45°) = \sin 330° \cos 45° + \cos 330° \sin 45°$$

$$= \left(-\frac{1}{2}\right) \cdot \left(\frac{\sqrt{2}}{2}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$$
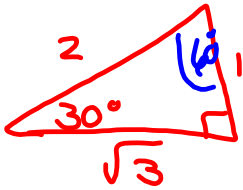
$$= \frac{-\sqrt{2}}{4} + \frac{\sqrt{6}}{4} =$$

$$= \boxed{\frac{\sqrt{6} - \sqrt{2}}{4}}$$

10. $\cos\left(\dfrac{\pi}{4} - \dfrac{\pi}{3}\right) = \cos\dfrac{\pi}{4}\cos\dfrac{\pi}{3} + \sin\dfrac{\pi}{4}\sin\dfrac{\pi}{3} =$

$$= \dfrac{\sqrt{2}}{2} \cdot \dfrac{1}{2} + \dfrac{\sqrt{2}}{2} \cdot \dfrac{\sqrt{3}}{2}$$

$$= \dfrac{\sqrt{2}}{4} + \dfrac{\sqrt{6}}{4} =$$

$$= \boxed{\dfrac{\sqrt{2} + \sqrt{6}}{4}}$$

$\sin 45° = \dfrac{1}{\sqrt{2}} \cdot \dfrac{\sqrt{2}}{\sqrt{2}}$

$= \dfrac{\sqrt{2}}{2}$

14. $\sin 167° \cos 107° - \cos 167° \sin 107°$

$$= \sin\left(167° - 107°\right) =$$

$$= \sin 60° =$$

$$= \boxed{\dfrac{\sqrt{3}}{2}}$$

20. $\sin x \cos 3x + \cos x \sin 3x$

$$= \sin(x + 3x)$$

$$= \boxed{\sin 4x}$$

34. Given $\sin \alpha = \dfrac{24}{25}$, $\alpha \in Q\,II$

$\cos \beta = \dfrac{-4}{5}$, $\beta \in Q\,III$

Find $\sin(\alpha-\beta)$, $\cos(\alpha-\beta)$, $\tan(\alpha-\beta)$ & determine the quadrant in which $\alpha-\beta$ lies.

$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$$= \dfrac{24}{25}\left(\dfrac{-4}{5}\right) - \left(\dfrac{-7}{25}\right)\left(\dfrac{3}{5}\right)$$

$$= \dfrac{-96}{125} - \dfrac{21}{125} = \boxed{\dfrac{-117}{125}}$$

$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$$= \left(\dfrac{-7}{25}\right)\left(\dfrac{-4}{5}\right) + \left(\dfrac{24}{25}\right)\left(\dfrac{-3}{5}\right)$$

$$= \dfrac{28}{125} - \dfrac{72}{125} = \boxed{\dfrac{-44}{125}}$$

$\tan(\alpha-\beta) = \dfrac{\sin(\alpha-\beta)}{\cos(\alpha-\beta)} = \dfrac{-117/125}{-44/125} = \boxed{\dfrac{117}{44}}$

$\alpha-\beta$ lies in $\boxed{\text{quadrant } III}$

$\begin{array}{r}{}^2 25 \\ \times 25 \\ \hline 125 \\ 500 \\ \hline 625\end{array}$ $\begin{array}{r} 24 \\ \times 24 \\ \hline 96 \\ 480 \\ \hline 576\end{array}$ $\begin{array}{r} {}^{6}{}^{11}625 \\ -576 \\ \hline 49\end{array}$

-7 25

-4 -3 5

Homework #6:

- 6.1 #1-69 odd (proofs)

- 6.2 #1-41 odd

- 6.3 #1-24; 30-36

& **memorize your identities!!!**