Homework hints/solutions:

$$3.\quad \frac{1}{2}\csc^2\frac{x}{2} = \csc^2 x + \cot x \csc x$$

Left-hand side $= \frac{1}{2}\left[\csc\left(\frac{x}{2}\right)\right]^2 =$

$$= \frac{1}{2}\left[\frac{1}{\sin\left(\frac{x}{2}\right)}\right]^2 = \frac{1}{2}\left[\frac{1}{\pm\sqrt{\frac{1-\cos x}{2}}}\right]^2 =$$

$$= \frac{1}{2}\left(\frac{1}{\frac{1-\cos x}{2}}\right) = \frac{1}{2}\cdot\frac{2}{1-\cos x} = \frac{1}{1-\cos x} =$$

$$= \frac{1}{1-\cos x}\cdot\frac{1+\cos x}{1+\cos x} = \frac{1+\cos x}{1-\cos^2 x} = \frac{1+\cos x}{\sin^2 x} =$$

$\sin^2 x + \cos^2 x = 1$

$$= \frac{1}{\sin^2 x} + \frac{\cos x}{\sin^2 x} = \csc^2 x + \frac{\cos x}{\sin x \sin x} =$$

$$= \csc^2 x + \frac{\cos x}{\sin x}\cdot\frac{1}{\sin x} =$$

$= \csc^2 x + \cot x \csc x =$ Right-hand side

$$4.\quad \sec 2x = \frac{\sec^2 x}{2 - \sec^2 x}$$

Left-hand side $= \frac{1}{\cos 2x} = \frac{1}{2\cos^2 x - 1} =$

$$= \frac{1}{2(\cos x)^2 - 1} = \frac{1}{2\left(\frac{1}{\sec x}\right)^2 - 1} =$$

$$= \frac{1}{\frac{2}{\sec^2 x} - 1\cdot\frac{\sec^2 x}{\sec^2 x}} = \frac{1}{\frac{2-\sec^2 x}{\sec^2 x}} =$$

$= 1\cdot\frac{\sec^2 x}{2-\sec^2 x} =$ Right-hand side ∎

$$10.\quad \cos 3x = 4\cos^3 x - 3\cos x$$

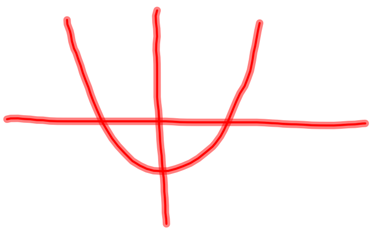
Left-hand side $= \cos(2x + x) =$

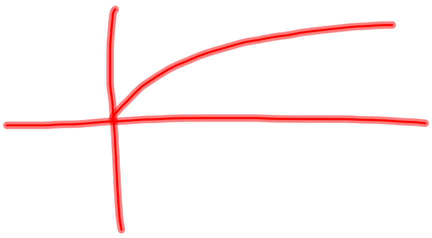
Inverse Trigonometric Functions

(6.4 book / 6.5 handout)

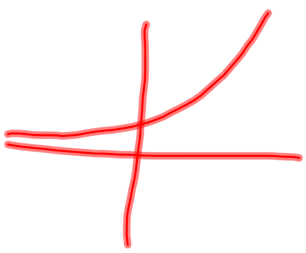
Recall from Algebra:

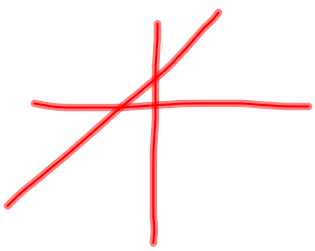
- f is a **function** if each input value (x) has a unique output $f(x)$.
  Examples: $f(x) = x^2 - 2$, $f(x) = \sqrt{x}$

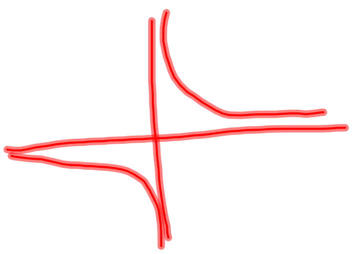


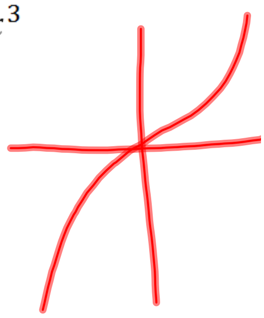


- f is **one-to-one** if, in addition, each y corresponds to only one x.
  Examples: $y = e^x$, $y = x + 3$, $y = \frac{1}{x}$, $y = x^3$









- If f is a one-to-one function, we can define its inverse $f^{-1}(x)$. Note that this notation is not exponentiation, i.e. $f^{-1}(x) \neq \frac{1}{f(x)}$

$$x^{-n} = \frac{1}{x^n}$$

- $f(x)$ and $g(x)$ are **inverses** if $(f \circ g)(x) = f\big(g(x)\big) = x = g\big(f(x)\big) = (g \circ f)(x)$, that is, **inverse functions "undo" each other**.

Example: $f(x) = x^3$, $g(x) = \sqrt[3]{x}$

$$(f \circ g)(x) = f(g(x)) = \left(\sqrt[3]{x}\right)^3 = x$$

$$(g \circ f)(x) = g(f(x)) = \sqrt[3]{x^3} = x$$

What do we mean by an Inverse Trig function?

Recall that **for a basic Trigonometric function**, e.g. $f(x) = \sin x$,

- **The input (x) is an angle**
- **The output $f(x)$ is a ratio of sides**

So **for an inverse Trigonometric function,**

- **The input (x) is a ratio of sides**
- **The output $f(x)$ is an angle**

Construction of the inverse of $f(x) = \sin x$:

$f(x) = x^3 - 8$

$y = x^3 - 8$

$x = y^3 - 8$

$x + 8 = y^3$

$y = \sqrt[3]{x+8}$

$f^{-1}(x) = \sqrt[3]{x+8}$

$y = \sin x$

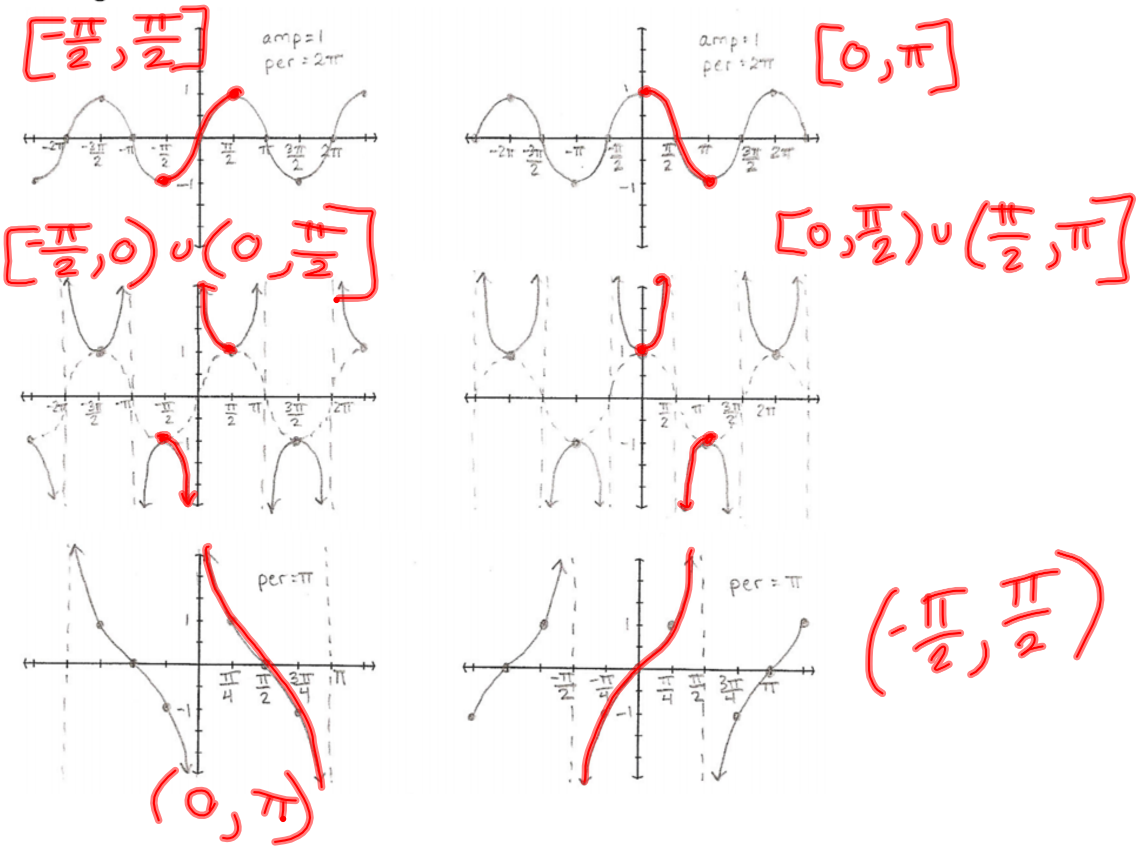
$x = \sin y$

$y =$ the angle whose sine value is x

$f^{-1}(x) = \sin^{-1} x = \arcsin x$

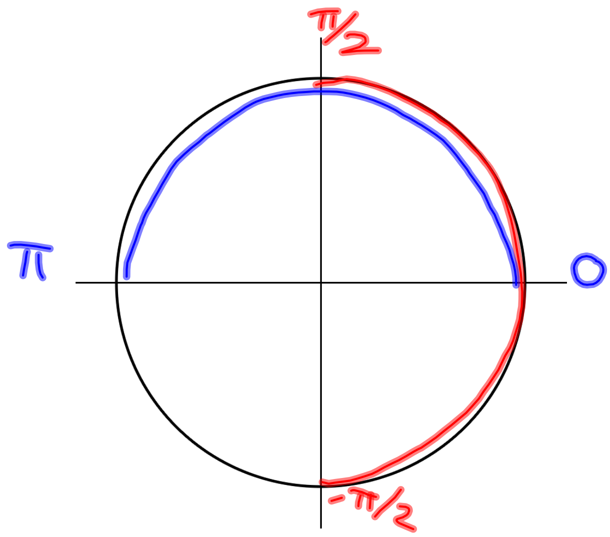
$\sin^{-1} x \neq \frac{1}{\sin x}$

But Trigonometric functions aren't one-to-one – how is the inverse defined? We must restrict the domain!

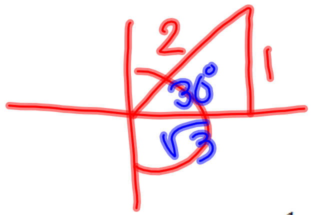


Summary of Restricted Domains:

Interval	Functions	Quadrants
$\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$	$\sin x, \csc x, \tan x$	IV & I
$(0,\pi)$	$\cos x, \sec x, \cot x$	I & II

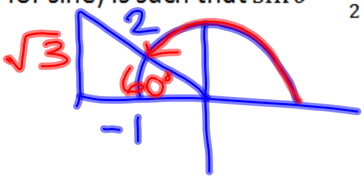


Evaluate the inverse trigonometric expression.



$\sin^{-1}\left(\frac{1}{2}\right) =$ $30° = \frac{\pi}{6}$

In words: What angle θ, between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ (the restricted domain for sine) is such that $\sin\theta = \frac{1}{2}$?



$\cos^{-1}\left(-\frac{1}{2}\right) =$ $120° = \frac{2\pi}{3}$

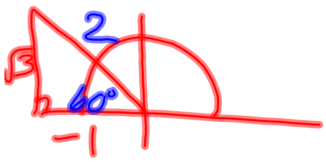
In words: What angle θ, between 0 and π (the restricted domain for cosine) is such that $\cos\theta = -\frac{1}{2}$?

$\tan^{-1}(-1) =$ $-45° = \frac{-\pi}{4}$

what angle θ in the restricted domain of tangent $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is such that $\tan\theta = -1$

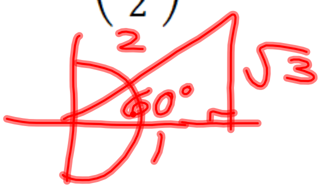
Evaluate.

$\cot^{-1}\left(-\frac{1}{\sqrt{3}}\right) =$ $120° = \frac{2\pi}{3}$

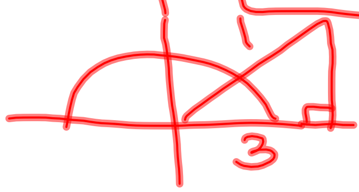


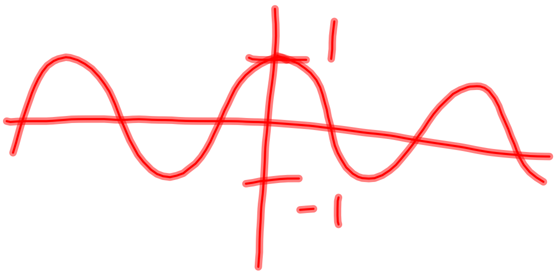
$\tan^{-1}(0) =$ $\boxed{0° = 0}$

$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) =$ $60° = \frac{\pi}{3}$



$\cos^{-1}(3) =$ $\boxed{\text{undefined}}$





$\csc^{-1}(-2) =$ $-30° = \frac{-\pi}{6}$

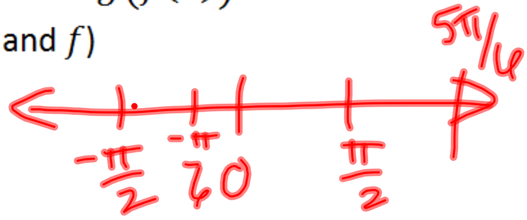
What happens when we compose a Trigonometric function with its inverse?

According to the definition,

$f(x)$ and $g(x)$ are inverses if $f\big(g(x)\big) = x$ and $g\big(f(x)\big) = x$
(for all x-values in the respective domains of g and f)

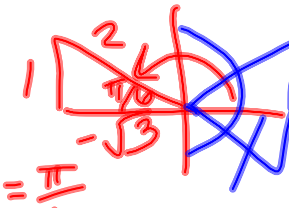
We would then expect

$\sin(\sin^{-1} x) = x$ and $\sin^{-1}(\sin x) = x$



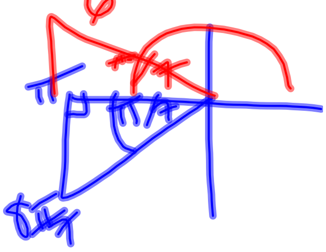
$\sin\left(\sin^{-1}\frac{1}{2}\right) = \sin 30^\circ = \frac{1}{2}$

$\sin^{-1}\left(\underbrace{\sin\left(\frac{5\pi}{6}\right)}\right) = \sin^{-1}\left(\frac{1}{2}\right) = 30^\circ = \frac{\pi}{6}$



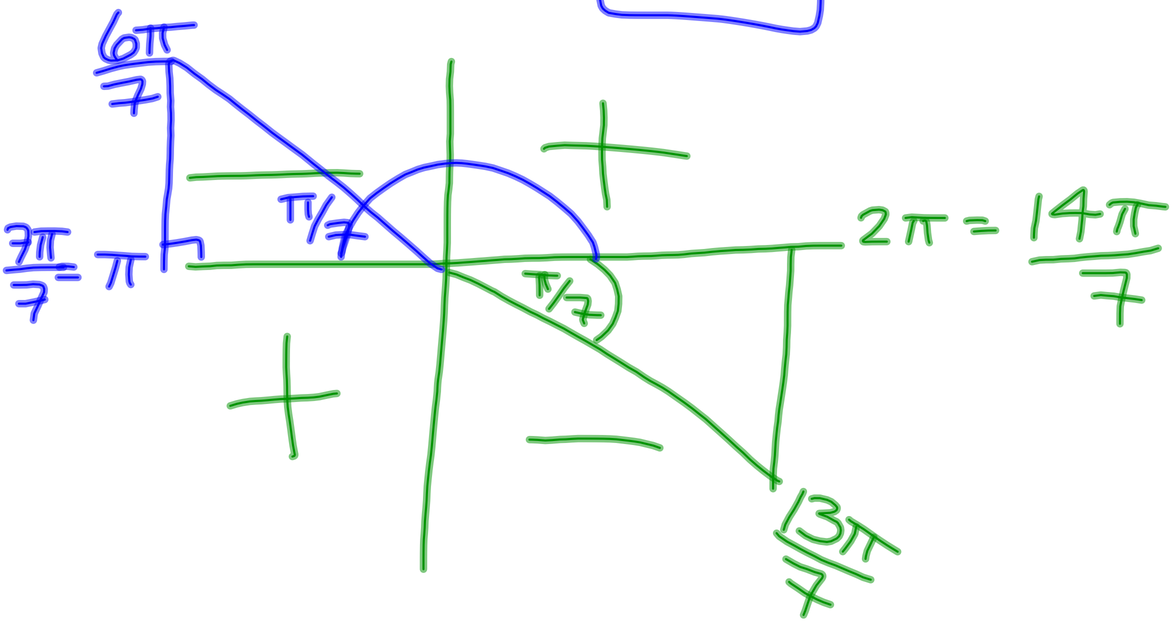
$\sin^{-1}\left(\underbrace{\sin\left(-\frac{\pi}{6}\right)}_{-\frac{1}{2}}\right) = -\frac{\pi}{6} = -30^\circ$

$\cos^{-1}\left(\cos\left(\frac{8\pi}{7}\right)\right) = \boxed{\frac{6\pi}{7}}$



$\sin(\underbrace{\sin^{-1} 3}_{\text{undef.}}) = \underline{\text{undefined}}$

$$\cot^{-1}\left(\cot\frac{13\pi}{7}\right) = \boxed{\frac{6\pi}{7}}$$



Homework:

6.5 Handout #1-24 all

Note the upcoming:

Quiz #6 - Wednesday, 04/17 (expect to prove an identity)
Quiz #7 - Tuesday, 04/23

Test #3 - Friday, 04/26
(if you will be missing the test for the WOW retreat,
make sure you schedule a time on Thurs, 04/25 to take it!)