# Chapter 6 - Trigonometric Identities and Equations

**Reciprocal Identities**

$\csc x = \frac{1}{\sin x}$ , $\sin x = \frac{1}{\csc x}$

$\sec x = \frac{1}{\cos x}$ , $\cos x = \frac{1}{\sec x}$

$\cot x = \frac{1}{\tan x}$ , $\tan x = \frac{1}{\cot x}$

**Ratio Identities**

$\tan x = \frac{\sin x}{\cos x}$ , $\cot x = \frac{\cos x}{\sin x}$

**Cofunction Identities**

$\sin\left(\frac{\pi}{2} - x\right) = \cos x$ , $\cos\left(\frac{\pi}{2} - x\right) = \sin x$

$\tan\left(\frac{\pi}{2} - x\right) = \cot x$ , $\cot\left(\frac{\pi}{2} - x\right) = \tan x$

$\csc\left(\frac{\pi}{2} - x\right) = \sec x$ , $\sec\left(\frac{\pi}{2} - x\right) = \csc x$

**Odd-Even Identities**

$\cos(-x) = \cos x$ , $\sin(-x) = -\sin x$ , $\tan(-x) = -\tan x$

$\sec(-x) = \sec x$ , $\csc(-x) = -\csc x$ , $\cot(-x) = -\cot x$

**Pythagorean Identities**

$\sin^2 x + \cos^2 x = 1$

$1 + \cot^2 x = \csc^2 x$

$\tan^2 x + 1 = \sec^2 x$

**Useful formulas from Algebra:**

$a^2 - b^2 = (a-b)(a+b)$

$(a+b)^2 = a^2 + 2ab + b^2$

$(a-b)^2 = a^2 - 2ab + b^2$

$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

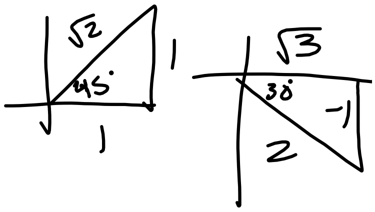
$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

$(a+b)^2 \neq a^2 + b^2$

$(a+b)(a+b)$

## 6.2 - Sum and Difference Identities

$$\sin(a+b) = \sin a \cos b + \cos a \sin b$$
$$\sin(a-b) = \sin a \cos b - \cos a \sin b$$
$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$
$$\cos(a-b) = \cos a \cos b + \sin a \sin b$$
$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$$
$$\tan(a-b) = \frac{\tan a - \tan b}{1 + \tan a \tan b}$$




$$\sin 375° = \sin(45° + 330°)$$

$$\sin(a+b) = \sin a \cos b + \cos a \sin b$$

$$\sin(45° + 330°) = \sin 45° \cos 330° + \cos 45° \sin 330°$$

$$= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\left(-\frac{1}{2}\right)$$

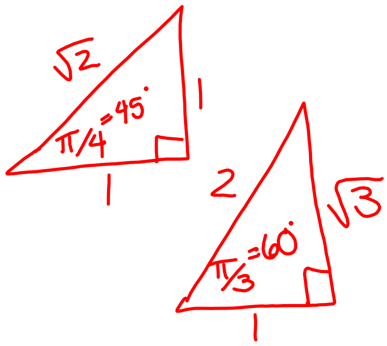
$$= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = \boxed{\frac{\sqrt{6} - \sqrt{2}}{4}}$$

$$\cos\left(\frac{\pi}{4} - \frac{\pi}{3}\right) = \cos\frac{\pi}{4}\cos\frac{\pi}{3} + \sin\frac{\pi}{4}\sin\frac{\pi}{3}$$

$$\cos(a - b) = \cos a \cos b + \sin a \sin b$$

$$= \frac{\sqrt{2}}{2} \cdot \frac{1}{2} + \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2}$$

$$= \boxed{\frac{\sqrt{2} + \sqrt{6}}{4}}$$



$$\sin \underset{a}{167^\circ} \cos \underset{b}{107^\circ} - \cos \underset{a}{167^\circ} \sin \underset{b}{107^\circ}$$

$$= \sin(167^\circ - 107^\circ)$$

$$= \sin 60^\circ$$

$$= \boxed{\sqrt{3}/2}$$

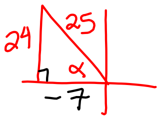
$$\sin \underset{a}{x} \cos \underset{b}{3x} + \cos \underset{a}{x} \sin \underset{b}{3x}$$

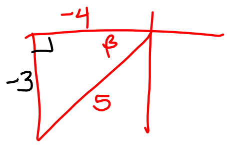
$$= \sin(x + 3x)$$

$$= \boxed{\sin 4x}$$

Given $\sin\alpha = \frac{24}{25}$, $\alpha \in Q\,II$

$\cos\beta = \frac{-4}{5}$, $\beta \in Q\,III$




Find $\sin(\alpha-\beta)$, $\cos(\alpha-\beta)$, $\tan(\alpha-\beta)$ & determine the quadrant in which $\alpha-\beta$ lies.

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
$$= \left(\frac{24}{25}\right)\left(\frac{-4}{5}\right) - \left(\frac{-7}{25}\right)\left(\frac{-3}{5}\right)$$
$$= \frac{-96}{125} - \frac{21}{125} = \boxed{\frac{-117}{125}}$$

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$
$$= \left(\frac{-7}{25}\right)\left(\frac{-4}{5}\right) + \left(\frac{24}{25}\right)\left(\frac{-3}{5}\right)$$
$$= \frac{28}{125} - \frac{72}{125} = \boxed{\frac{-44}{125}}$$

$$\tan(\alpha-\beta) = \frac{\sin(\alpha-\beta)}{\cos(\alpha-\beta)} = \frac{-117/125}{-44/125} = \boxed{\frac{117}{44}}$$

$\alpha-\beta \in Q\,III$ because sin & cos are neg & tan is positive

Given $\cos\alpha = \frac{8}{17}$, $\alpha \in Q\,IV$

$\sin\beta = \frac{-24}{25}$, $\beta \in Q\,III$

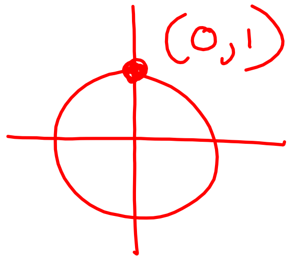
find $\sin(\alpha+\beta)$, $\cos(\alpha+\beta)$, $\tan(\alpha+\beta)$, & determine the quadrant in which $\alpha+\beta$ lies.

*Pythagorean triples that are useful to know:

3, 4, 5 ; 5, 12, 13 ; 7, 24, 25 ;

& 8, 15, 17

## Cofunction Identities




The function of an angle is equal to the cofunction of its complement.

$\theta$ & $90° - \theta$ or $\theta$ & $\frac{\pi}{2} - \theta$
are complementary angles

$\cos\left(\frac{\pi}{2} - x\right) = \cos\frac{\pi}{2}\cos x + \sin\frac{\pi}{2}\sin x$

$= 0 \cdot \cos x + 1 \cdot \sin x$

$= \boxed{\sin x}$

## Double-Angle Identities

$\sin 2\theta = \sin(\theta + \theta)$

$= \sin\theta\cos\theta + \cos\theta\sin\theta$

$\boxed{\sin 2\theta = 2(\sin\theta)(\cos\theta)}$

$\sin(42x) = \sin[2(21x)] = \boxed{2\sin 21x\cos 21x}$

$\sin(8x) = \sin[2(4x)] = \boxed{2\sin 4x\cos 4x}$

$$\boxed{\sin 2\theta = 2\sin\theta\cos\theta}$$

The sine of twice any angle is equal to two times the sine of that angle times the cosine of that angle.

$\sin 6\theta = 2\sin 3\theta\cos 3\theta$

$\sin 18\theta = 2\sin 9\theta\cos 9\theta$

$\sin 44\theta = 2\sin 22\theta\cos 22\theta$

$\sin 3\theta \neq 2\sin\frac{3\theta}{2}\cos\frac{3\theta}{2}$

$\to \sin(2\theta+\theta)$

$= \sin 2\theta\cos\theta + \cos 2\theta\sin\theta$

$$\begin{aligned}\cos 2\theta &= \cos(\theta+\theta)\\ &= \cos\theta\cos\theta - \sin\theta\sin\theta\\ &= (\cos\theta)^2 - (\sin\theta)^2\end{aligned}$$

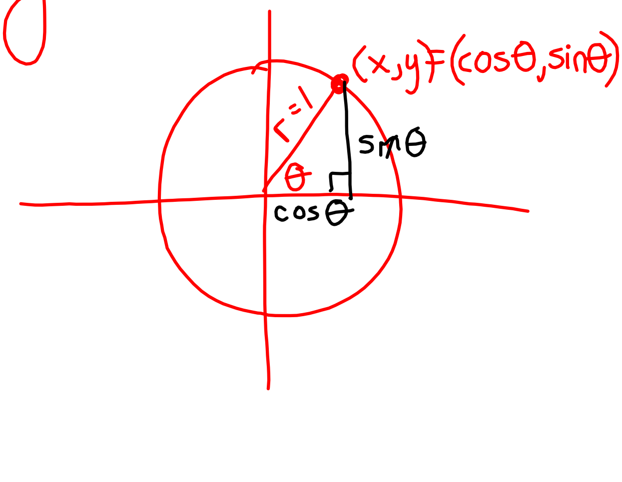
$$\boxed{\cos 2\theta = \cos^2\theta - \sin^2\theta}$$

$$[f(x)]^2 = f^2(x) \neq f(x)^2 = f(x^2)$$

Pythagorean Identity

$$(\sin\theta)^2 + (\cos\theta)^2 = 1^2$$

$$\boxed{\sin^2\theta + \cos^2\theta = 1}$$



The other two Pythagorean Identities are derived from the first.

$$\frac{\sin^2 x + \cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$$

$$\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$\boxed{\tan^2 x + 1 = \sec^2 x}$$

$$\frac{\sin^2 x + \cos^2 x}{\sin^2 x} = \frac{1}{\sin^2 x}$$

$$\boxed{1 + \cot^2 x = \csc^2 x}$$