4.1 Inverse Functions

Recall:

f is a function if each input value (x) has exactly one output $f(x)$

Functions pass the vertical line test.

f is a one-to-one function if, in addition, each y corresponds to only one x.

One-to-one functions pass both the horizontal line test and the vertical line test.

Formally, a function is one-to-one if different inputs have ~~the same~~ different outputs, i.e.

$if\ a \neq b, then\ f(a) \neq f(b),$

Or equivalently, f is one-to-one if when the outputs are the same, the inputs are the same, i.e.

$if\ f(a) = f(b), then\ a = b.$

Proving that a function *is* one-to-one v. proving that a function *is not* one-to-one (problems 17-24 from section 4.1)

To show that f(x) is not one-to-one, it is enough to provide a single counter-example, i.e. 2 different inputs that yield the same output

$f(x) = x^2 - 5$

$\cancel{a^2 - 5 = b^2 - 5}$

$\cancel{a^2 = b^2}$

$\cancel{\pm a = \pm b}$

$f(1) = (1)^2 - 5 = 1 - 5 = -4$

$f(-1) = (-1)^2 - 5 = 1 - 5 = -4$

Since $f(1) = f(-1)$, f is not 1-1

To show that f(x) <u>is</u> one-to-one, we must prove it in general.

$$f(x) = -2x^3 + 1$$

$$f(a) = f(b)$$

$$-2a^3 + 1 = -2b^3 + 1$$

$$-1 \qquad -1$$

$$\frac{-2a^3}{-2} = \frac{-2b^3}{-2}$$

$$\sqrt[3]{a^3} = \sqrt[3]{b^3}$$

$$a = b$$

Since $f(a) = f(b)$ implies that $a = b$, f is one-to-one!

If a function is one-to-one, then it has an inverse.

Interchanging the first and second coordinates of each ordered pair in a relation produces the inverse function.

$$f(x) = \{(1,2), (3,4), (5,6), (7,8)\}$$

$$f^{-1}(x) = \{(2,1), (4,3), (6,5), (8,7)\}$$

If a relation is defined by an equation, interchanging the variables produces an equation of the inverse relation.

$$y = -2x^3 + 1$$

$$x = -2y^3 + 1$$

The domain of a one-to-one function f is the range of the inverse f^{-1}.

The range of a one-to-one function f is the domain of the inverse f^{-1}.

$$f^{-1}(x) \neq \frac{1}{f(x)}$$

Obtaining the formula for an inverse:

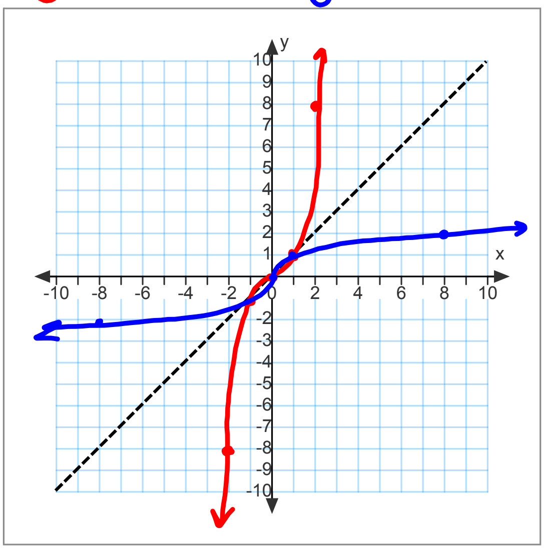
1. Replace $f(x)$ with y
2. Interchange x and y
3. Solve for y
4. Replace y with $f^{-1}(x)$

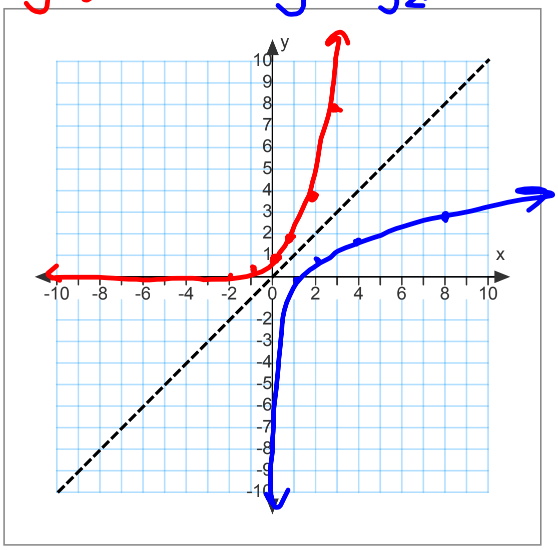
$$f(x) = -2x^3 + 1$$
$$y = -2x^3 + 1$$
$$x = -2y^3 + 1$$
$$x - 1 = -2y^3$$
$$\frac{x-1}{-2} = y^3$$
$$\sqrt[3]{\frac{x-1}{-2}} = y$$

$$\boxed{f^{-1}(x) = \sqrt[3]{\frac{x-1}{-2}}}$$

The graph of f^{-1} is a reflection of the graph of f across the line $y = x$

$y = x^3$ $\quad$ $y = \sqrt[3]{x}$ $\qquad$ $y = 2^x$ $\quad$ $y = \log_2 x$





If f & g are inverses, then

$(f \circ g)(x) = x$ for all x in the domain of g

AND

$(g \circ f)(x) = x$ for all x in the domain of f

86. $f(x) = \sqrt[3]{x+4}$; $f^{-1}(x) = x^3 - 4$

use composition to show f^{-1} is as given.

$$(f \circ f^{-1})(x) = \sqrt[3]{(x^3-4)+4} = \sqrt[3]{x^3} = x \checkmark$$

$$(f^{-1} \circ f)(x) = \left(\sqrt[3]{x+4}\right)^3 - 4 = x+4-4 = x \checkmark$$

88. $f(x)=\dfrac{x+6}{3x-4}$, $f^{-1}(x)=\dfrac{4x+6}{3x-1}$

$(f\circ f^{-1})(x) = \dfrac{\left(\frac{4x+6}{3x-1}\right)+6}{3\left(\frac{4x+6}{3x-1}\right)-4} = \dfrac{\frac{4x+6}{3x-1}+\frac{6(3x-1)}{3x-1}}{\frac{3(4x+6)}{3x-1}-\frac{4(3x-1)}{3x-1}} =$

$= \dfrac{\frac{4x+6+18x-6}{3x-1}}{\frac{12x+18-12x+4}{3x-1}} = \dfrac{\cancel{22}x}{\cancel{3x-1}} \cdot \dfrac{\cancel{3x-1}}{\cancel{22}} = x$ ✓

$(f^{-1}\circ f)(x) = \dfrac{4\left(\frac{x+6}{3x-4}\right)+6}{3\left(\frac{x+6}{3x-4}\right)-1} = \dfrac{\frac{4(x+6)}{3x-4}+\frac{6(3x-4)}{3x-4}}{\frac{3(x+6)}{3x-4}-\frac{1(3x-4)}{3x-4}} =$

$= \dfrac{\frac{4x+24+18x-24}{3x-4}}{\frac{3x+18-3x+4}{3x-4}} = \dfrac{\cancel{22}x}{\cancel{3x-4}} \cdot \dfrac{\cancel{3x-4}}{\cancel{22}} = x$ ✓

4.2 Exponential Functions

$f(x) = a^x$

base exponent

$a > 0$ excludes $(-1)^{1/2}$

$a \neq 1$ excludes $1^x = 1$ constant

Note: the variable is in the exponent, unlike power functions/polynomials

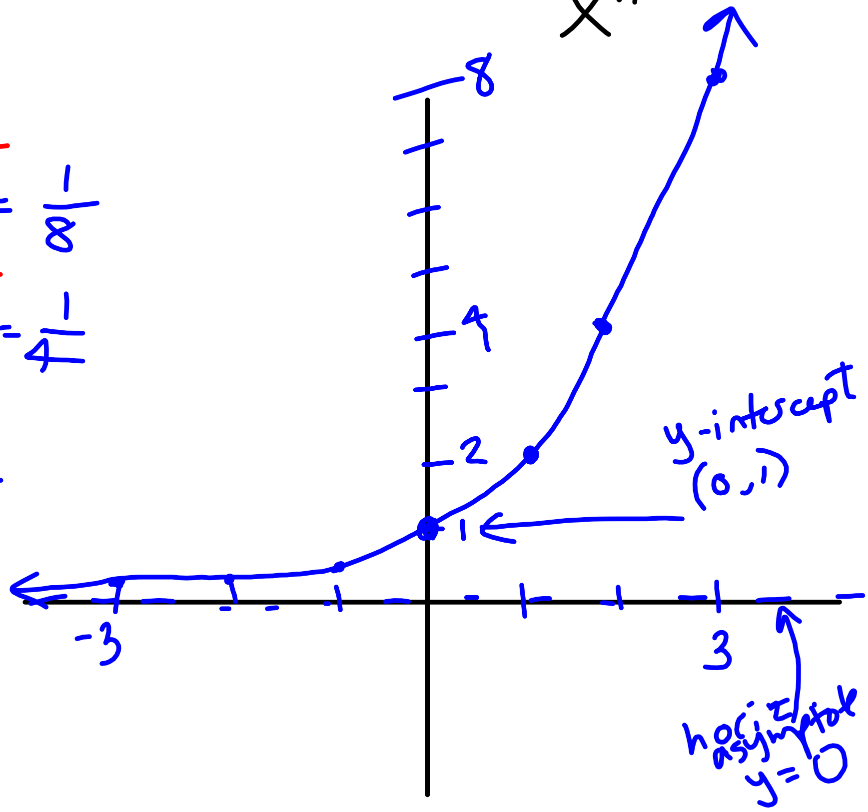
$f(x) = x^2$ power/poly

$f(x) = 2^x$ exp.

$$f(x) = 2^x$$

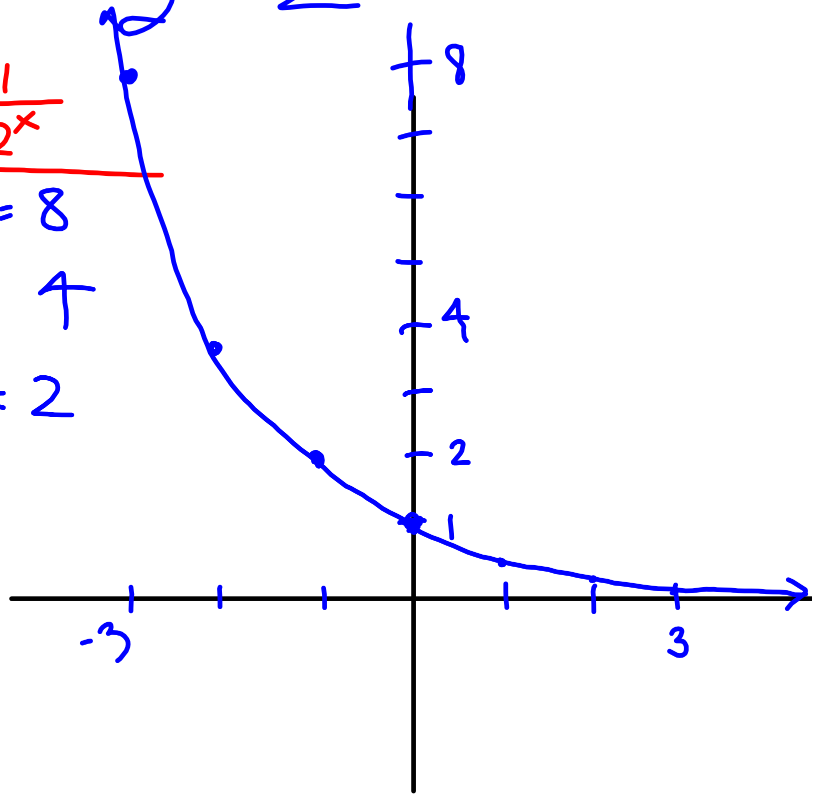
$$x^{-n} = \frac{1}{x^n}$$

x	2^x
-3	$2^{-3} = \frac{1}{2^3} = \frac{1}{8}$
-2	$2^{-2} = \frac{1}{2^2} = \frac{1}{4}$
-1	$2^{-1} = \frac{1}{2}$
0	$2^0 = 1$
1	$2^1 = 2$
2	$2^2 = 4$
3	$2^3 = 8$



$$f(x) = \left(\frac{1}{2}\right)^x = \frac{1}{2^x} = 2^{-x}$$

x	$\left(\frac{1}{2}\right)^x = \frac{1}{2^x}$
-3	$\frac{1}{2^{-3}} = 2^3 = 8$
-2	$\frac{1}{2^{-2}} = 2^2 = 4$
-1	$\frac{1}{2^{-1}} = 2^1 = 2$
0	$\frac{1}{2^0} = 1$
1	$\frac{1}{2^1} = \frac{1}{2}$
2	$\frac{1}{2^2} = \frac{1}{4}$
3	$\frac{1}{2^3} = \frac{1}{8}$



Homework #7

3.7	#23-37 odd	variation and applications
4.1	#17-23 odd	prove f is one-to-one; prove g is not one-to-one
	#59-63 odd	determine if f is on-to-one and if so, determine its inverse
	#77-81 odd	sketch the inverse function by reflecting over y=x
	#83-87 odd	use composition to show that the functions are inverses
4.2	#5-10all	match an exponential function to its graph
	#11-41odd	sketch graphs of exponential functions using transformations
	#43a,b,c,45,47	compound interest word problems
4.3	#1-8all	sketch graphs of logarithmic functions
	#9-33odd	evaluate log expressions without a calculator
	#35-53 odd	convert between logarithmic and exponential expressions
	#69-77 odd	apply change of base formula & calculator to approximate log expressions
	#83-90 all	graph logarithmic functions using transformations