8.3 Systems of Equations & Matrices

Substitution method:

$x + 3y = 5 \Rightarrow x = -3y + 5$

$2x - 7y = 3$

$2(-3y+5) - 7y = 3$

$-6y + 10 - 7y = 3$

$-13y = -7$

$y = 7/13$

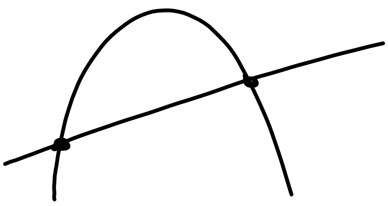
$x = -3\left(\frac{7}{13}\right) + 5$

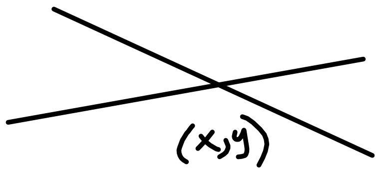
$= \frac{-21}{13} + \frac{65}{13}$

$x = \frac{44}{13}$

$\boxed{\left(\frac{44}{13}, \frac{7}{13}\right)}$

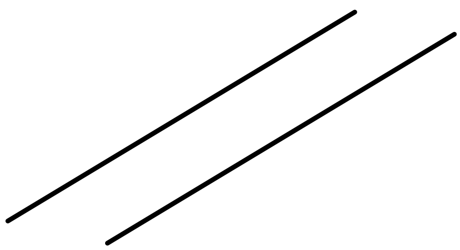
In general, the solution to a system of equations is the set of all points where the graphs of those equations coincide (or intersect).

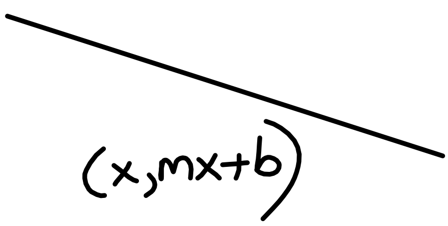




The graphs of two linear equations can intersect either:

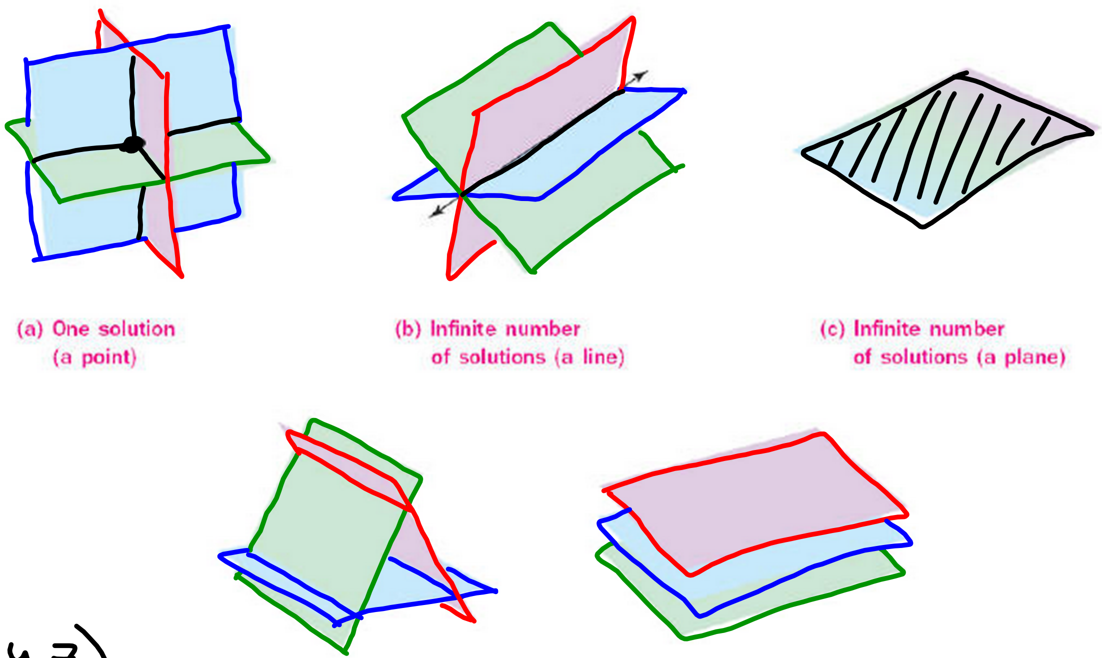
- at a single point (x,y) if their slopes are different,
- not at all if they are parallel, or
- at infinitely many points if the two equations correspond to the same graph.





$$\begin{cases} 2x+4y-z=7 \\ x-y+5z=2 \\ 3x+2y+z=-3 \end{cases}$$

The graph of a linear equation in 3 variables is a plane in 3-dimensional space. Three planes can intersect in essentially 5 different ways:



(a) One solution (a point)

(b) Infinite number of solutions (a line)

(c) Infinite number of solutions (a plane)

(d) No solution

(e) No solution

(x,y,z)

16. $\begin{cases} 2x+y=1 \\ 3x+2y=-2 \end{cases} \Rightarrow \left[\begin{array}{cc|c} 2 & 1 & 1 \\ 3 & 2 & -2 \end{array}\right]$

augmented matrix

We will rewrite a system of linear equations given in **standard form** (constant on right, variables on left in order) as an **augmented matrix**, then use a process called **Gauss-Jordan Elimination** to transform that augmented matrix into a matrix in "**reduced row-echelon form**" (1's along the diagonal from upper left to bottom right, 0's above and below), so that we can read the solution to the system directly from the right hand side.

$1x+0y=a$
$0x+1y=b$

$\left[\begin{array}{cc|c} 1 & 0 & a \\ 0 & 1 & b \end{array}\right]$ Solution: (a, b)

Rules of (Gaussian) elimination/addition method:

We can:

- multiply any row by a non-zero constant
- add any non-zero multiple of one row to another
- interchange any two rows

Note that these are the same rules you used to solve systems of linear equations in two variables, except with the word "row" instead of the word "equation."

Note the resources available at asms.net/brewer under "Precal":

"Matrices" - a guide on "How to Solve a System of Equations in Three Variables Using an Augmented Matrix "

"Blank Matrix Guide" - a template for one method of Gaussian Elimination that always works

16. $$\begin{cases}2x+y=1\\3x+2y=-2\end{cases} \Longrightarrow \left[\begin{array}{cc|c}2&1&1\\3&2&-2\end{array}\right]$$

interchange R_1 & R_2

$$\left[\begin{array}{cc|c}3&2&-2\\2&1&1\end{array}\right] \xrightarrow{R_1+(-1)R_2} \left[\begin{array}{cc|c}1&1&-3\\2&1&1\end{array}\right]$$

$$\xrightarrow{R_2+(-2)R_1} \left[\begin{array}{cc|c}1&1&-3\\0&-1&7\end{array}\right] \xrightarrow{R_2(-1)} \left[\begin{array}{cc|c}1&1&-3\\0&1&-7\end{array}\right]$$

$1+(-2)(1)$

$1+(-2)(-3)$

$$\xrightarrow{R_1+(-1)R_2} \left[\begin{array}{cc|c}1&0&4\\0&1&-7\end{array}\right] \Rightarrow \begin{array}{l}x=4\\y=-7\end{array}$$

$-3-(-7)$

$$\boxed{(4,-7)}$$

28. $$\begin{aligned}x-y+2z&=0\\x-2y+3z&=-1\\2x-2y+z&=-3\end{aligned}$$

$$\left[\begin{array}{ccc|c}1&-1&2&0\\1&-2&3&-1\\2&-2&1&-3\end{array}\right] \xrightarrow{\textit{swap rows}} \left[\begin{array}{ccc|c}1&&&\\&&&\\&&&\end{array}\right]$$

or multiply R_1 by constant

$-2+(-1)(-1)=-1$

$3+(-1)(2)=1$

$-1+(-1)(0)=-1$

$-2+(-2)(-1)=0$

$1+(-2)(2)=-3$

$-3+(-2)(0)=-3$

R2+ (−1)·R1

R3+ (−2)·R1

$$\longrightarrow \left[\begin{array}{ccc|c}1&-1&2&0\\0&-1&1&-1\\0&0&-3&-3\end{array}\right] \xrightarrow{R2\cdot(-1)} \left[\begin{array}{ccc|c}1&-1&2&0\\0&1&-1&1\\0&0&-3&3\end{array}\right]$$

R1+ (1)·R2

~~*R3+ ( )·R2*~~

$$\longrightarrow \left[\begin{array}{ccc|c}1&0&1&1\\0&1&-1&1\\0&0&-3&3\end{array}\right] \xrightarrow{R3\cdot\left(-\frac{1}{3}\right)} \left[\begin{array}{ccc|c}1&0&1&1\\0&1&-1&1\\0&0&1&-1\end{array}\right]$$

R1 + (−1)·R3

R2 + (1)·R3

$$\longrightarrow \left[\begin{array}{ccc|c}1&0&0&2\\0&1&0&0\\0&0&1&-1\end{array}\right]$$

$\boxed{\textit{solution: } (2, 0, -1)}$

32.

$$\begin{cases} 2x-3y+2z=2 \\ x+4y-z=9 \\ -3x+y-5z=5 \end{cases}$$

$$\left[\begin{array}{ccc|c} 2 & -3 & 2 & 2 \\ 1 & 4 & -1 & 9 \\ -3 & 1 & -5 & 5 \end{array}\right] \xrightarrow[R_1 \,\&\, R_2]{\textit{swap rows}} \left[\begin{array}{ccc|c} 1 & 4 & -1 & 9 \\ 2 & -3 & 2 & 2 \\ -3 & 1 & -5 & 5 \end{array}\right]$$

$$-3+(-2)(4)=-11 \qquad 1+3(4)=13$$
$$2+(-2)(-1)=4 \qquad -5+3(-1)=-8$$
$$2+(-2)(9)=-16 \qquad 5+3(9)=32$$

$$\xrightarrow[R3+(3)\cdot R1]{R2+(-2)\cdot R1} \left[\begin{array}{ccc|c} 1 & 4 & -1 & 9 \\ 0 & -11 & 4 & -16 \\ 0 & 13 & -8 & 32 \end{array}\right] \xrightarrow{R2\cdot\left(-\frac{1}{11}\right)} \left[\begin{array}{ccc|c} 1 & 4 & -1 & 9 \\ 0 & 1 & -\frac{4}{11} & \frac{16}{11} \\ 0 & 13 & -8 & 32 \end{array}\right]$$

$$-1+(-4)\left(-\frac{4}{11}\right)=-\frac{11}{11}+\frac{16}{11}=\frac{5}{11} \qquad -8+(-13)\left(-\frac{4}{11}\right)=-\frac{88}{11}+\frac{52}{11}$$
$$9+(-4)\left(\frac{16}{11}\right)=\frac{99}{11}-\frac{64}{11}=\frac{35}{11} \qquad 32+(-13)\left(\frac{16}{11}\right)=\frac{352}{11}-\frac{208}{11}=\frac{144}{11}$$

$$\xrightarrow[R3+(-13)\cdot R2]{R1+(-4)\cdot R2} \left[\begin{array}{ccc|c} 1 & 0 & \frac{5}{11} & \frac{35}{11} \\ 0 & 1 & -\frac{4}{11} & \frac{16}{11} \\ 0 & 0 & -\frac{36}{11} & \frac{144}{11} \end{array}\right] \xrightarrow{R3\cdot\left(-\frac{11}{36}\right)} \left[\begin{array}{ccc|c} 1 & 0 & \frac{5}{11} & \frac{35}{11} \\ 0 & 1 & -\frac{4}{11} & \frac{16}{11} \\ 0 & 0 & 1 & -4 \end{array}\right]$$

$$\frac{144}{11}\cdot\frac{-11}{36}=$$

$$\frac{35}{11}+\left(-\frac{5}{11}\right)(-4)=\frac{55}{11}=5$$
$$\frac{16}{11}+\left(\frac{4}{11}\right)(-4)=0$$

$$\xrightarrow[R2+\left(\frac{4}{11}\right)\cdot R3]{R1+\left(-\frac{5}{11}\right)\cdot R3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -4 \end{array}\right]$$

$\boxed{\textit{solution: } (5, 0, -4)}$

HW #9 (due Fri, Oct 17)

solve systems of equations in 2 and 3 variables using a matrix to get a single, integer solution:

- 8.1 #17, 19, 21 (but solve using a matrix)
- 8.3 #27, 31
- 8.2 #3*,7,9 (but solve using a matrix)

*some books have a typo on #3

the third equation should read:

2x+y+z=-3