The Derivative

The slope of the tangent line to the graph of f at the point $(c, f(c))$ is given by:

$$m = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x}$$

The derivative of f at x is given by

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

20. $f(x) = x^3 + x^2$

find the derivative

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$f(x) = x^3 + x^2$

$f(x+h) = (x+h)^3 + (x+h)^2$

			1		$(a+b)^0$
		1	1		$(a+b)^1$
	1	2	1		$(a+b)^2$
1	3	3	1		$(a+b)^3$
1	4	6	4	1	$(a+b)^4$

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^3 + (x+h)^2 - (x^3 + x^2)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 + \cancel{x^2} + 2xh + h^2 - \cancel{x^3} - \cancel{x^2}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(3x^2 + 3xh + h^2 + 2x + h)}{\cancel{h}} = \boxed{3x^2 + 2x}$$

2.1 Differentiability & Continuity

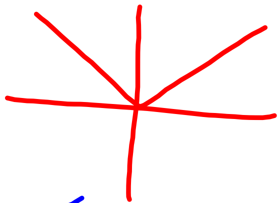
Alternative definition of the derivative at the point $(c, f(c))$:

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

All differentiable functions are continuous, but
not all continuous functions are differentiable.

e.g. $f(x) = |x|$

$$f'(0) = \lim_{x \to 0} \frac{|x| - |0|}{x - 0} = \lim_{x \to 0} \frac{|x|}{x}$$



$$\frac{|x|}{x} = \begin{cases} \frac{x}{x} = 1, & x > 0 \\ \frac{-x}{x} = -1, & x < 0 \end{cases}$$

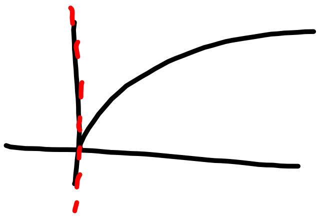
$\lim_{x \to 0^+} |x| = 1$; $\lim_{x \to 0^-} |x| = -1$

f is not differentiable @ x=0.

$f(x) = |x + 3|$

$\lim_{x \to -3}$

$f(x) = \sqrt{x}$



$$\lim_{x\to 0^+} \frac{\sqrt{x}-\sqrt{0}}{x-0} = \lim_{x\to 0^+} \frac{\sqrt{x}}{x}$$

$$= \lim_{x\to 0^+} \frac{x^{1/2}}{x^1} = \lim_{x\to 0^+} \frac{1}{x^{1/2}}$$

$$= \lim_{x\to 0^+} \frac{1}{\sqrt{x}} = \infty$$

$f'(x)$ does not exist for $x = 0$.

$\Rightarrow$ vertical tangent line

$$\frac{1}{1/2} = 2;\ \frac{1}{1/100} = 100;\ \frac{1}{1/100000} = 100000 \to \infty$$

$$\frac{x^m}{x^n} = \frac{x^{m-n}}{1} = \frac{1}{x^{n-m}}$$

2.2 Basic Differentiation Rules

1. The derivative of a constant function is zero, i.e.,

for $c \in \mathbb{R}$, $\quad \frac{d}{dx}[c] = 0$

Proof: $f(x) = c$

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{c-c}{h} = \lim_{h\to 0} \frac{0}{h} = \lim_{h\to 0} 0$$

$$= \boxed{0}$$

$$\frac{d}{dx}[7] = 0$$

$$[327\pi^2]' = 0$$

2. Power Rule for $n \epsilon \mathbb{Q}$, $\frac{d}{dx}[x^n] = nx^{n-1}$

$\binom{n}{k} = \frac{n!}{k!(n-k)!}$ "n choose k"

Proof: $f(x) = x^n$

$n! = n(n-1)\cdot(n-2)\cdots 3\cdot 2\cdot 1$

Recall the binomial expansion:

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2 + \cdots + \frac{n!}{k!\,(n-k)!}a^{n-k}b^k + \cdots + b^n$$

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^n - \cancel{x^n}}{h} = \lim_{h\to 0} \frac{\cancel{x^n} + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \ldots + h^n - \cancel{x^n}}{}$$

$$= \lim_{h\to 0} \frac{\cancel{h}\left(nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \ldots + h^{n-1}\right)}{\cancel{h}}$$

$$= nx^{n-1} + 0 + 0 + \ldots + 0$$

$$= \boxed{nx^{n-1}}$$

$[x^1]' = 1x^0 = 1\cdot 1 = 1$

Special case: $\frac{d}{dx}[x] = 1$

Examples:

$$\frac{d}{dx}[x^7] = 7x^6$$

$$\frac{d}{dx}[\pi^3] = 0$$

$$\frac{d}{dx}[2e] = 0$$

$$\frac{d}{dx}[\sqrt{x}] = \frac{d}{dx}[x^{1/2}] = \boxed{\frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}}$$

$$\frac{d}{dx}\left[\frac{1}{x^3}\right] = \frac{d}{dx}[x^{-3}] = \boxed{-3x^{-4} = \frac{-3}{x^4}}$$

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3. Constant Multiple Rule $\in \mathbb{R}$, $\frac{d}{dx}[cf(x)] = cf'(x)$

4. Sum & Difference Rules $\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$

Examples:

$f(x) = 3x^2$

$f'(x) = 3 \cdot [x^2]' = 3 \cdot 2x = \boxed{6x}$

$f(x) = \frac{3}{x} = 3x^{-1}$

$f'(x) = 3 \cdot [x^{-1}]' = 3 \cdot [-x^{-2}] = \boxed{-3x^{-2}}$

$g(x) = 2x^3 - x^2 + 3x$

$g'(x) = \boxed{6x^2 - 2x + 3}$

$y = 4x^{3/2} - 5x^4 + 2x^{\frac{1}{3}} - 7$

$y' = \boxed{6x^{1/2} - 20x^3 + \frac{2}{3}x^{-2/3}}$

Derivatives of Trig Functions

1. $\frac{d}{dx}[\sin x] = \cos x$
2. $\frac{d}{dx}[\cos x] = -\sin x$
3. $\frac{d}{dx}[\tan x] = \sec^2 x$
4. $\frac{d}{dx}[\cot x] = -\csc^2 x$
5. $\frac{d}{dx}[\sec x] = \sec x \tan x$
6. $\frac{d}{dx}[\csc x] = -\csc x \cot x$

Proof that (sin x)'=cos x

$$(\sin x)' = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} =$$

$$= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \lim_{h\to 0} \left[\frac{\cos x \sin h}{h} - \frac{\sin x - \sin x \cos h}{h} \right]$$

$$= \lim_{h\to 0} \left[\cos x \cdot \underbrace{\frac{\sin h}{h}}_{1} - \sin x \cdot \underbrace{\frac{(1-\cos h)}{h}}_{0} \right]$$

$$= \boxed{\cos x}$$

Homework #4:

Find the derivative by the limit process:
2.1 #1-23 odd

Use the alternate form to find the derivative:
2.1 #61-69 odd

Describe the x-values where the function is differentiable (given a graph):
2.1 #71-79 odd

Find the derivative using the basic derivative rules we have learned so far:
2.2 #3-51 odd

Work through intuitive exercises on **Khan Academy**:
- Slope of secant lines
- Tangent lope is limiting value of secant slope
- Derivative intuition
- Visualizing derivatives
- Graphs of functions and their derivatives
- The formal and alternate form of the derivative
- Derivatives 1
- Recognizing slopes of curves
- Power rule
- Special derivatives