Evaluating Limits Analytically

Basic Limits

Let $b, c \in \mathbb{R}$, $n > 0$ *an integer*, $f, g -$ *functions*, $\lim_{x\to c} f(x) = L$, $\lim_{x\to c} g(x) = K$

1. Constant $\lim_{x\to c} b = b$
2. Identity $\lim_{x\to c} x = c$
3. Polynomial $\lim_{x\to c} x^n = c^n$
4. Scalar Multiple $\lim_{x\to c}[bf(x)] = bL$
5. Sum or Difference • $\lim_{x\to c}[f(x) \pm g(x)] = L \pm K$
6. Product • $\lim_{x\to c}[f(x)g(x)] = LK$
7. Quotient • $\lim_{x\to c}\left[\frac{f(x)}{g(x)}\right] = \frac{L}{K}$, $K \neq 0$
8. Power • $\lim_{x\to c}[f(x)]^n = L^n$

$\lim_{x\to c}[f(x) \pm g(x)] = \lim_{x\to c} f(x) \pm \lim_{x\to c} g(x)$

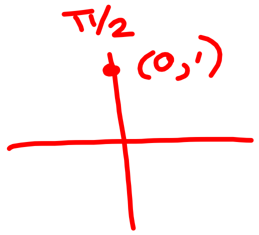
$\lim_{x\to c}[f(x) g(x)] = \left[\lim_{x\to c} f(x)\right] \cdot \left[\lim_{x\to c} g(x)\right]$

$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$, $\lim_{x\to c} g(x) \neq 0$

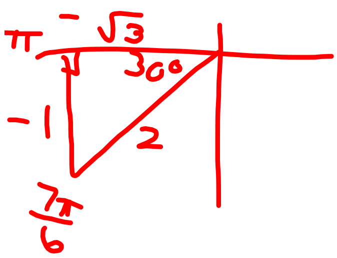
$\lim_{x\to c}[f(x)]^n = \left[\lim_{x\to c} f(x)\right]^n$

Note: If substitution yields $\frac{0}{0}$, an indeterminate form, the expression must be rewritten in order to evaluate the limit.

30. $\lim_{x\to 1} \sin \frac{\pi x}{2} = \sin \frac{\pi}{2} = \boxed{1}$



36. $\lim_{x\to 7} \sec\left(\frac{\pi x}{6}\right) = \sec \frac{7\pi}{6} = \boxed{-\frac{2}{\sqrt{3}}}$



38. $\lim\limits_{x \to c} f(x) = \frac{3}{2}$; $\lim\limits_{x \to c} g(x) = \frac{1}{2}$

(a) $\lim\limits_{x \to c} \left[4 f(x)\right] = 4 \cdot \lim\limits_{x \to c} f(x) = 4 \cdot \frac{3}{2} = \boxed{6}$

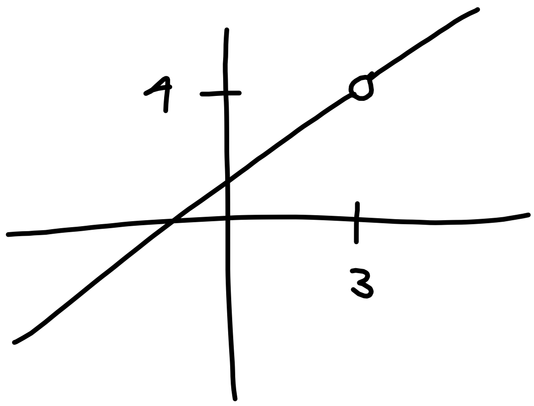
(b) $\lim\limits_{x \to c} \left[f(x) + g(x)\right] = \lim\limits_{x \to c} f(x) + \lim\limits_{x \to c} g(x) = \frac{3}{2} + \frac{1}{2} = \boxed{2}$

(c) $\lim\limits_{x \to c} \left[f(x) g(x)\right] = \left(\lim\limits_{x \to c} f(x)\right)\left(\lim\limits_{x \to c} g(x)\right) = \frac{3}{2} \cdot \frac{1}{2} = \boxed{\frac{3}{4}}$

(d) $\lim\limits_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x \to c} f(x)}{\lim\limits_{x \to c} g(x)} = \frac{3/2}{1/2} = \frac{3}{2} \cdot \frac{2}{1} = \boxed{3}$

$$\lim_{x \to 3} \frac{x^2 - 2x - 3}{x - 3} = \lim_{x \to 3} \frac{(x-3)(x+1)}{x-3}$$

$$= \lim_{x \to 3} (x+1) = \boxed{4}$$



$$\lim_{x\to 4} \frac{x-4}{\sqrt{x}-2} \cdot \frac{\sqrt{x}+2}{\sqrt{x}+2}$$

$$= \lim_{x\to 4} \frac{(x-4)(\sqrt{x}+2)}{x-4}$$

$$= \lim_{x\to 4} (\sqrt{x}+2) = \sqrt{4}+2 = 2+2 = \boxed{4}$$

Given $f(x) = 2x^2 + 3x + 1$

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$\lim_{h\to 0} \frac{2(x+h)^2+3(x+h)+1 - (2x^2+3x+1)}{h}$$

$$= \lim_{h\to 0} \frac{2(x^2+2xh+h^2)+3x+3h+1-2x^2-3x-1}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{2x^2}+4xh+2h^2+\cancel{3x}+3h\cancel{+1}-\cancel{2x^2}-\cancel{3x}\cancel{-1}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(4x+2h+3)}{\cancel{h}} = 4x+2(0)+3 = \boxed{4x+3}$$

$$f(x) = x^3$$

Pascal's Δ

$(a+b)^0 = 1$

$(a+b)^1 = 1a + 1b$

$(a+b)^2 = 1a^2 + 2ab + 1b^2$

$(a+b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3$

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$\lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h\to 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 \cancel{- x^3}}{h} =$$

$$= \lim_{h\to 0} \frac{\cancel{h}(3x^2 + 3xh + h^2)}{\cancel{h}} = 3x^2 + 3x(0) + (0)^2 = \boxed{3x^2}$$

1.3 Evaluating Limits Analytically

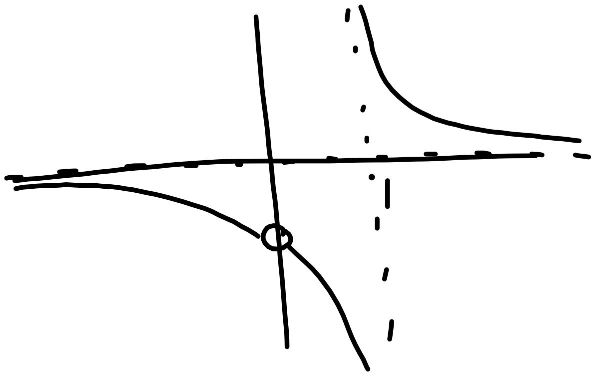
42. $h(x) = \dfrac{x^2 - 3x}{x} = \dfrac{x\,(x-3)}{x} = x - 3$

(a) $\lim\limits_{x\to -2} h(x) = \dfrac{(-2)^2 - 3(-2)}{-2} = \dfrac{4+6}{-2} = \dfrac{10}{-2} = \boxed{-5}$

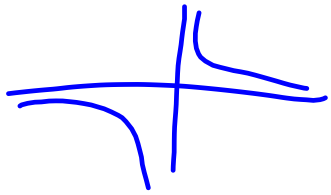
(b) $\lim\limits_{x\to 0} h(x) = 0 - 3 = \boxed{-3}$

44. $\lim_{x \to 1} \frac{x}{x^2 - x} = \lim_{x \to 1} \frac{x}{x(x-1)} = \lim_{x \to 1} \frac{1}{x-1}$

$\boxed{\text{undefined / does not exist}}$

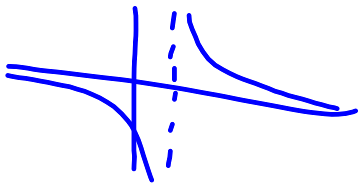


$y = \frac{1}{x}$



$f(x-1)$

$y = \frac{1}{x-1}$



48. $\lim_{x \to -1} \frac{x^3 + 1}{x + 1} = \lim_{x \to -1} \frac{\cancel{(x+1)}(x^2 - x + 1)}{\cancel{x+1}}$

$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

$= \lim_{x \to -1} (x^2 - x + 1) = (-1)^2 - (-1) + 1 = 1 + 1 + 1$

$= \boxed{3}$

54. $\lim_{x\to 0} \dfrac{\sqrt{2+x}-\sqrt{2}}{x}$

$~~\sqrt{a+b} = \sqrt{a} + \sqrt{b}~~$

$= \lim_{x\to 0} \dfrac{\sqrt{2+x}-\sqrt{2}}{x} \cdot \dfrac{\sqrt{2+x}+\sqrt{2}}{\sqrt{2+x}+\sqrt{2}}$

$= \lim_{x\to 0} \dfrac{(2+x)-2}{x(\sqrt{2+x}+\sqrt{2})}$

$= \lim_{x\to 0} \dfrac{\cancel{x}^{1}}{\cancel{x}(\sqrt{2+x}+\sqrt{2})}$

$= \lim_{x\to 0} \dfrac{1}{\sqrt{2+x}+\sqrt{2}} = \dfrac{1}{\sqrt{2}+\sqrt{2}} = \dfrac{1}{2\sqrt{2}} \cdot \dfrac{\sqrt{2}}{\sqrt{2}}$

$= \boxed{\dfrac{\sqrt{2}}{4}}$

$a+b$
$a-b$
$b+a$
$b-a$
$(a-b)(a+b) = a^2-b^2$

58. $\lim_{x\to 0} \dfrac{\frac{1}{x+4}-\frac{1}{4}}{x} = \lim_{x\to 0} \dfrac{\frac{1}{x+4}\cdot\frac{4}{4} - \frac{1}{4}\cdot\frac{x+4}{x+4}}{\frac{x}{1}}$

$4-x-4$

$= \lim_{x\to 0} \dfrac{\left(\frac{4-(x+4)}{4(x+4)}\right)}{\frac{x}{1}} = \lim_{x\to 0} \left(\dfrac{-\cancel{x}}{4(x+4)}\right)\cdot \dfrac{1}{\cancel{x}}$

$= \lim_{x\to 0} \dfrac{-1}{4(x+4)} = \dfrac{-1}{4(0+4)} = \boxed{\dfrac{-1}{16}}$

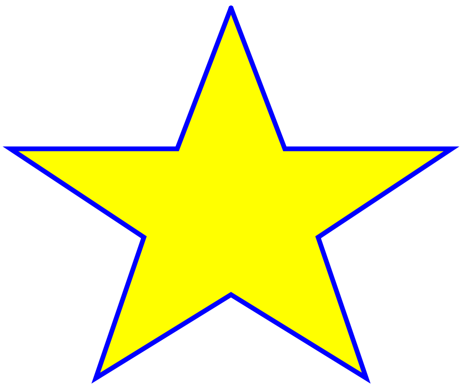
66. $\displaystyle \lim_{x\to 2} \frac{x^5 - 32}{x - 2}$

$$\begin{array}{c|cccccc} 2 & 1x^5 & 0x^4 & 0x^3 & 0x^2 & 0x & -32 \\ & & 2 & 4 & 8 & 16 & 32 \\ \hline & 1 & 2 & 4 & 8 & 16 & \boxed{0} \end{array}$$

$$\lim_{x\to 2} \frac{\cancel{(x-2)}(x^4+2x^3+4x^2+8x+16)}{\cancel{x-2}}$$

$$= 2^4 + 2(2)^3 + 4(2)^2 + 8(2) + 16$$

$$= 16(5) = \boxed{80}$$



HW #1 (submitted 11/7):

- 1.2 #1-7odd,9-18all

HW #2 (due 11/14):

- **1.2 #23, 25, 27, 29, 30, 31** **epsilon delta**
  (and watch all of the Khan Academy epsilon-delta videos!)
- **1.3 #11,17,27-35odd, 39-61odd** **evaluating limits analytically**
- **1.3 #67-77odd; 87, 88** **limits with trig, squeeze theorem**
  ***problems in red are NOT listed on syllabus**

When to have 1st quiz?