$$\lim_{x \to 3} \frac{x-3}{2-\sqrt{x+1}} \cdot \frac{2+\sqrt{x+1}}{2+\sqrt{x+1}}$$

$$= \lim_{x \to 3} \frac{(x-3)(2+\sqrt{x+1})}{(2-\sqrt{x+1})(2+\sqrt{x+1})} \longleftarrow \begin{array}{c}(a-b)(a+b)\\ a^2-b^2\end{array}$$

$$= \lim_{x \to 3} \frac{(x-3)(2+\sqrt{x+1})}{\underbrace{4-(x+1)}_{3-x}} = \lim_{x \to 3} \frac{\cancel{(x-3)}(2+\sqrt{x+1})}{-\cancel{(x-3)}}$$

$$= \boxed{-4}$$

exp	conj
$a+\sqrt{b}$	$a-\sqrt{b}$
$\sqrt{a}+b$	$\sqrt{a}-b$
$a-\sqrt{b}$	$a+\sqrt{b}$
$\sqrt{a}-b$	$\sqrt{a}+b$
$a+\sqrt{b+c}$	$a-\sqrt{b+c}$
$\sqrt{a-b}+c$	$\sqrt{a-b}-c$
$\sqrt{a}-\sqrt{b}$	$\sqrt{a}+\sqrt{b}$

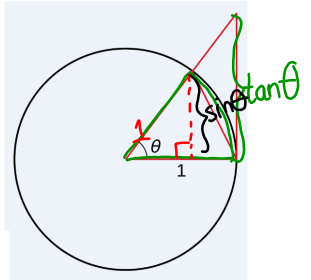
1.3 The Squeeze Theorem

$$\lim_{x\to 0}\frac{\sin x}{x} = ?$$

Area of whole circle $= \pi r^2|_{r=1} = \pi$

$$\frac{\text{Area of whole circle}}{\text{Total angle of circle}} = \frac{\text{Area of sector}}{\theta}$$

$$\frac{\pi}{2\pi} = \frac{\text{Area of sector}}{\theta} \;\rightarrow\; \text{Area of sector} = \frac{\theta}{2}$$



Area of outer triangle ≥ Area of sector ≥ Area of inner triangle

$$\frac{\tan\theta}{2} \quad\geq\quad \frac{\theta}{2} \quad\geq\quad \frac{\sin\theta}{2}$$

Multiply through by $\frac{2}{\sin\theta}$

$$\frac{\sin\theta}{2\cos\theta}\cdot\frac{2}{\sin\theta} \quad\geq\quad \frac{\theta}{2}\cdot\frac{2}{\sin\theta} \quad\geq\quad \frac{\sin\theta}{2}\cdot\frac{2}{\sin\theta}$$

$$\frac{1}{\cos\theta} \quad\geq\quad \frac{\theta}{\sin\theta} \quad\geq\quad 1$$

Take reciprocals and reverse inequalities

$$\cos\theta \quad\leq\quad \frac{\sin\theta}{\theta} \quad\leq\quad 1$$

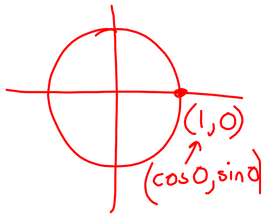
Take limits

$$\lim_{\theta\to 0}\cos\theta \quad\leq\quad \lim_{\theta\to 0}\frac{\sin\theta}{\theta} \quad\leq\quad \lim_{\theta\to 0} 1$$

$\cos 0$

$$1 \leq \lim_{\theta\to 0}\frac{\sin\theta}{\theta} \leq 1$$

$$\Rightarrow \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$$



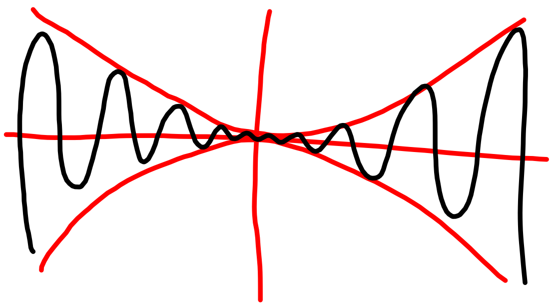
The Squeeze Theorem:

If $f(x) \leq g(x) \leq h(x)$ *and* $\lim_{x\to c} f(x) = L = \lim_{x\to c} h(x)$,

Then $\lim_{x\to c} g(x) = L$.

$$-1 \leq \sin x \leq 1$$

$$-x^2 \leq x^2\sin x \leq x^2$$



Special Limits Derived by Squeeze Theorem:

$$\boxed{\lim_{x\to 0}\frac{\sin x}{x} = 1} \;;\; \boxed{\lim_{x\to 0}\frac{1-\cos x}{x} = 0}$$

Memorize!!

Use the squeeze theorem to find $-1 \le \cos\theta \le 1$

$$\lim_{x\to 0}\left(x^2\cos\frac{5}{x}-3\right)$$

$$-1 \le \cos\frac{5}{x} \le 1$$

$$-x^2 \le x^2\cos\frac{5}{x} \le x^2$$

$$-x^2-3 \le x^2\cos\frac{5}{x}-3 \le x^2-3$$

$$\lim_{x\to 0}(-x^2-3) \le \lim_{x\to 0}\left(x^2\cos\frac{5}{x}-3\right) \le \lim_{x\to 0}(x^2-3)$$

$$-3 \le \lim_{x\to 0}\left(x^2\cos\frac{5}{x}-3\right) \le -3$$

Therefore, by the Squeeze Theorem

$$\lim_{x\to 0}\left(x^2\cos\frac{5}{x}-3\right) = \boxed{-3}$$

68. $$\lim_{x\to 0}\frac{3(1-\cos x)}{x}$$

$$\lim_{x\to 0}\frac{\sin x}{x} = 1$$

$$\lim_{x\to 0}\frac{1-\cos x}{x} = 0$$

$$= \left(\lim_{x\to 0} 3\right)\left(\lim_{x\to 0}\frac{1-\cos x}{x}\right)$$

$$= 3 \cdot 0$$

$$= \boxed{0}$$

72. $\lim\limits_{x\to 0} \frac{\tan^2 x}{x}$

$$= \lim_{x\to 0} \frac{\frac{\sin^2 x}{\cos^2 x}}{\frac{x}{1}}$$

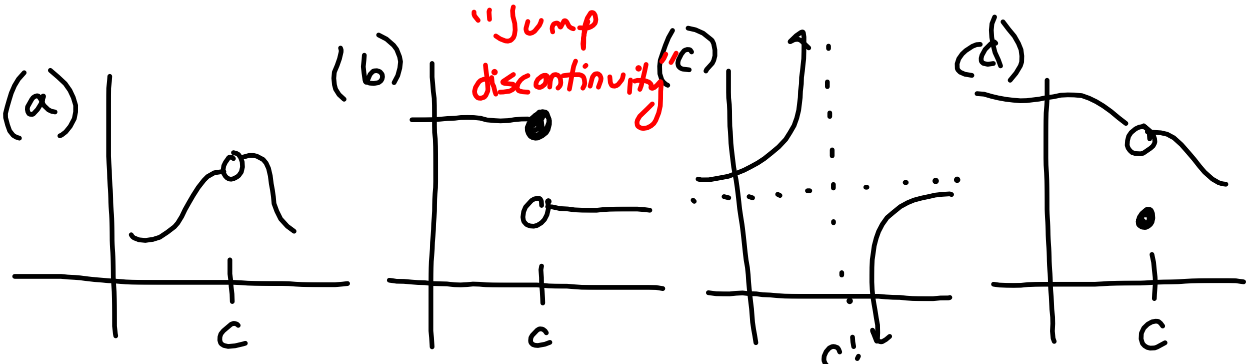
$$= \lim_{x\to 0} \frac{\sin x \sin x}{\cos x \cos x} \cdot \frac{1}{x}$$

$$= \lim_{x\to 0} \left(\frac{\sin x}{x}\right)\left(\frac{\sin x}{\cos^2 x}\right)$$

$$= 1 \cdot 0 = \boxed{0}$$

78. $\lim\limits_{x\to 0} \frac{\sin 2x}{\sin 3x}$

1.4 Continuity and One-Sided Limits

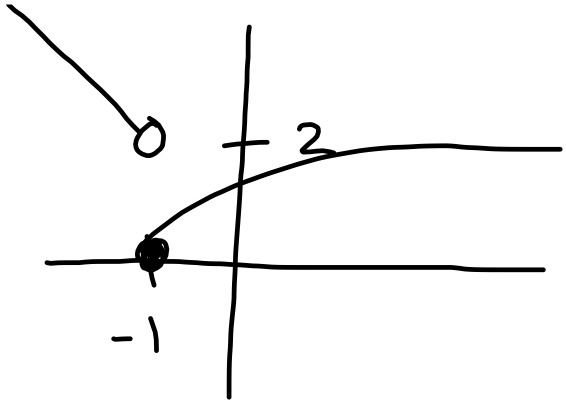


f(c) is undefined

$\lim\limits_{x\to c} f(x)$ does not exist

$\lim\limits_{x\to c} f(x) \neq f(c)$

These are all discontinuities
(a) and (d) are removable
(b) and (c) are nonremovable



$$\lim_{x \to -1^-} f(x) = 2$$

$$\lim_{x \to -1^+} f(x) = 0$$

$$\lim_{x \to -1} f(x) = DNE$$

One-Sided Limits

$\lim_{x \to c^+} f(x) = L$ *limit from the right*

$\lim_{x \to c^-} f(x) = L$ *limit from the left*

$$\boxed{\lim_{x \to c} f(x) = L \text{ if and only if } \lim_{x \to c^-} f(x) = L = \lim_{x \to c^+} f(x)}$$

Continuity at a point

A function f is <u>continuous at c</u> if the following 3 conditions are met:

1. $f(c)$ is defined
2. Limit of $f(x)$ exists when x approaches c
3. Limit of $f(x)$ when x approaches c is equal to $f(c)$

$f(x)$ is continuous at c if

$$\lim_{x \to c} f(x) = f(c)$$

Continuity on an open interval

A function is <u>continuous on an open interval</u> if it is continuous at each point in the interval. A function that is continuous on the entire real line $(-\infty, \infty)$is <u>everywhere continuous</u>.

Continuity on a closed interval

A function f is <u>continuous on the closed interval</u> $[a, b]$ if it is continuous on the open interval I (a, b) and $\lim_{x \to a^+} f(x) = f(a)$ and $\lim_{x \to b^-} f(x) = f(b)$.