$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

$f'(x)$ "f prime of x"

$\frac{dy}{dx} = \frac{d}{dx}(y)$ "derivative of y with respect to x"

y' "y prime"

$\frac{d}{dx}[f(x)]$ "the derivative with respect to x of $f(x)$"

$D_x[y]$ "the partial derivative with respect to x of y"

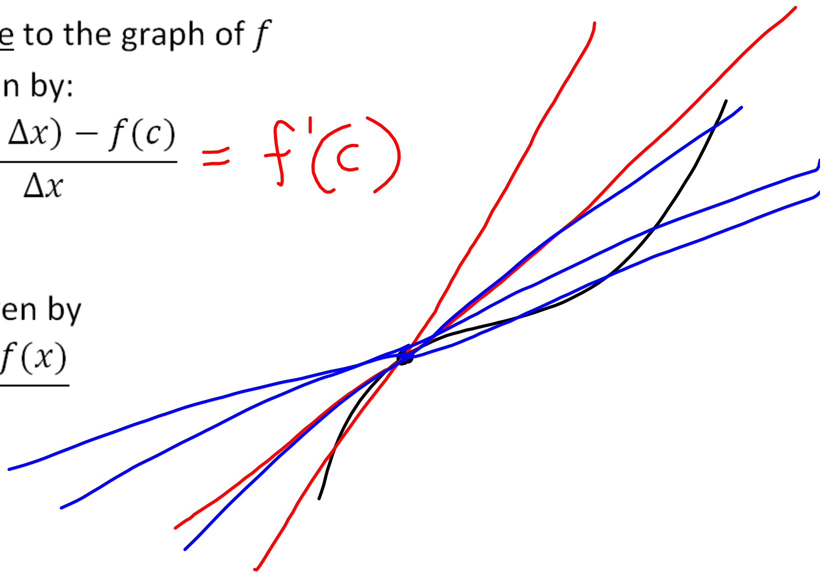
__The Derivative__

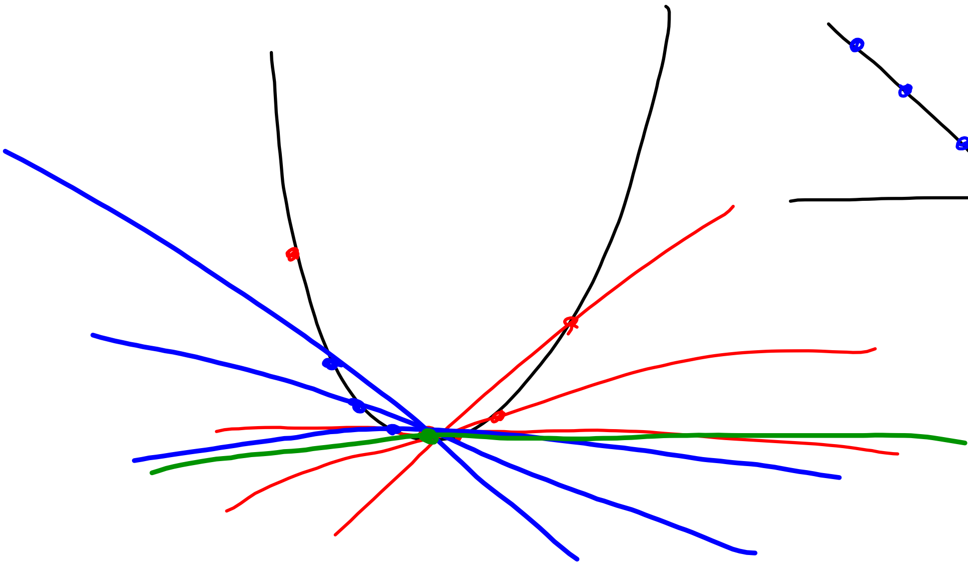
The __slope of the tangent line__ to the graph of f at the point $(c, f(c))$ is given by:

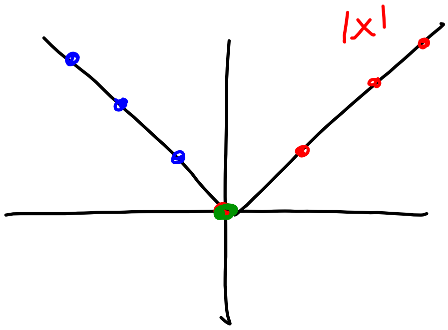
$$m = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x} = f'(c)$$

The __derivative of f at x__ is given by

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$



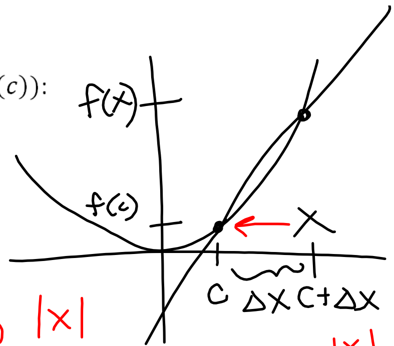




2.1 Differentiability & Continuity

Alternative definition of the derivative at the point $(c, f(c))$:

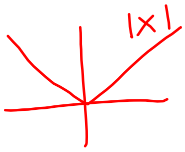
$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$



All differentiable functions are continuous, but not all continuous functions are differentiable.

e.g. $f(x) = |x|$

$$f'(0) = \lim_{x \to 0} \frac{|x| - |0|}{x - 0} = \lim_{x \to 0} \frac{|x|}{x}$$



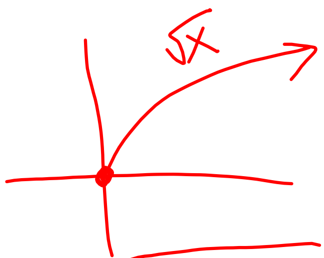
$$\lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} \frac{-x}{x} = \lim_{x \to 0^-} (-1) = -1$$

$$\lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} \frac{x}{x} = \lim_{x \to 0^+} (1) = 1$$

Because the left- & right-handed limits are unequal, the limit in general does not exist @ $x=0$, hence the derivative defined by that limit does not exist (i.e. $|x|$ is not differentiable @ 0)

$f(x) = |x + 3|$

$f(x) = \sqrt{x}$ continuous on $[0, \infty)$



$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

$$f'(0) = \lim_{x \to 0^+} \frac{\sqrt{x} - \sqrt{0}}{x - 0} = \lim_{x \to 0^+} \frac{\sqrt{x}}{x}$$

$$= \lim_{x \to 0^+} \frac{x^{1/2}}{x^1} = \lim_{x \to 0^+} \frac{1}{x^{1/2}} = \lim_{x \to 0^+} \frac{1}{\sqrt{x}}$$

$$\frac{x^m}{x^n} = \frac{x^{m-n}}{1} = \frac{1}{x^{n-m}}$$

$\frac{1}{1/10}, \frac{1}{1/1000}, \frac{1}{1/1000000}$ $= \infty = f'(0)$

vertical tangent line

∞ limit DNE $\Rightarrow f'(0)$ is undefined

$\sqrt{x}$ is not differentiable @ 0

2.2 Basic Differentiation Rules

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

1. The derivative of a constant function is zero, i.e.,

for $c \in \mathbb{R}$, $\frac{d}{dx}[c] = 0$

Proof:

$$[c]' = \frac{d}{dx}[c] = \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} \frac{0}{h}$$

$$= \lim_{h \to 0} 0 = 0$$

2. Power Rule for $n \epsilon \mathbb{Q}$, $\frac{d}{dx}[x^n] = nx^{n-1}$

Special case: $\frac{d}{dx}[x] = 1$

$$[x^1]' = 1 \cdot x^0 = 1 \cdot 1 = 1$$

Proof:

Recall the binomial expansion:

$$n! = n(n-1)(n-2)(n-3)\cdots 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2 + \cdots + \frac{n!}{k!\,(n-k)!}a^{n-k}b^k + \cdots + b^n$$

$$\frac{d}{dx}[x^n] = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h} =$$

$$= \lim_{h \to 0} \frac{\cancel{x^n} + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + h^n - \cancel{x^n}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}\left(nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}\underset{\to 0}{h} + \cdots + \underset{\to 0}{h^{n-1}}\right)}{\cancel{h}}$$

$$= \boxed{nx^{n-1}}$$

Examples:

$$x^{-1/2} = \frac{1}{x^{1/2}} = \frac{1}{\sqrt{x}}$$

$$\frac{d}{dx}[x^7] = 7x^6$$

$$\frac{d}{dx}[\pi^3] = 0$$

$$\frac{d}{dx}[2e] = 0$$

$$\frac{d}{dx}[\sqrt{x}] = \frac{d}{dx}\left[x^{1/2}\right] = \frac{1}{2}x^{\frac{1}{2}-1} = \boxed{\frac{1}{2}x^{-1/2}} = \boxed{\frac{1}{2\sqrt{x}}}$$

$$\frac{d}{dx}\left[\frac{1}{x^3}\right] = \frac{d}{dx}\left[x^{-3}\right] = \boxed{-3x^{-4}} = \boxed{\frac{-3}{x^4}}$$

2.2 Basic Differentiation Rules

1. The <u>derivative of a constant function</u> is zero, i.e.,

for $c \in \mathbb{R}$, $\frac{d}{dx}[c] = 0$

2. <u>Power Rule</u> for $n \epsilon \mathbb{Q}$, $\frac{d}{dx}[x^n] = nx^{n-1}$

3. <u>Constant Multiple Rule</u> $\in \mathbb{R}$, $\frac{d}{dx}[cf(x)] = cf'(x)$

4. <u>Sum & Difference Rules</u> $\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$

Examples:

$f(x) = 3x^2$

$f'(x) = 3(x^2)' = 3(2x) = \boxed{6x}$

$f(x) = \frac{3}{x} = 3x^{-1}$

$f'(x) = \boxed{-3x^{-2}} = \boxed{-\frac{3}{x^2}}$

$g(x) = 2x^3 - x^2 + 3x$

$g'(x) = \boxed{6x^2 - 2x + 3}$

$y = 4x^{3/2} - 5x^4 + 2x^{\frac{1}{3}} - 7$

$y' = \boxed{6x^{1/2} - 20x^3 + \frac{2}{3}x^{-2/3}}$

Derivatives of Trig Functions

1. $\frac{d}{dx}[\sin x] = \cos x$
2. $\frac{d}{dx}[\cos x] = -\sin x$
3. $\frac{d}{dx}[\tan x] = \sec^2 x$
4. $\frac{d}{dx}[\cot x] = -\csc^2 x$
5. $\frac{d}{dx}[\sec x] = \sec x \tan x$
6. $\frac{d}{dx}[\csc x] = -\csc x \cot x$