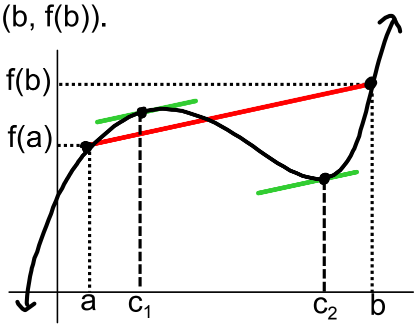
3.2 Rolle's Theorem & The Mean Value Theorem

The Mean Value Thereom (MVT) states: If f is continuous on [a,b] and differentiable on (a,b), then there exists at least one c in (a,b) such that the slope of the tangent line at c is equal to the slope of the secant line through (a, f(a)) and (b, f(b)).



If f is continuous on [a,b]

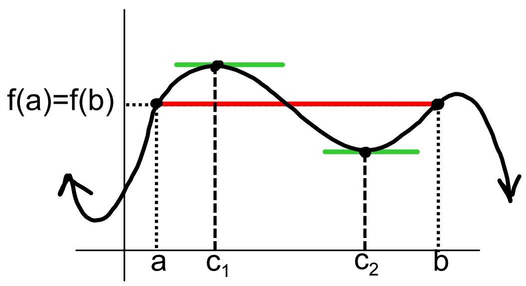
[$f(c)$ exists, $\lim_{x\to c} f(x)$ exists, & $\lim_{x\to c} f(x) = f(c)$ $\forall c \in [a,b]$]

and differentiable on (a,b)

[$f'(c)$ exists $\forall c \in (a,b)$]

then there exists at least one $c \in (a,b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$

Rolle's Theorem is a special case of the MVT where f(a)=f(b), (and hence involving horizontal secant/tangent lines)



If f is continuous on [a,b], differentable on (a,b) and $f(a) = f(b)$

[If $f(a)=f(b)$, $\frac{f(b)-f(a)}{b-a} = 0$]

$\exists\ c \in (a,b)$ such that $f'(c) = 0$.

$\exists$ ← there exists

Note that neither the Mean Value Theorem nor Rolle's Theorem apply to the following functions on the given intervals:

$$f(x) = \frac{x+5}{x-2}, \qquad [1,3]$$

f is not continuous on [1,3].

$$g(x) = |x-2|, \qquad [1,3]$$

g is continuous on [1,3], but not differentiable on (1,3).

Can Rolle's Theorem be applied?
If so, find all guaranteed values of c in (a,b).

8. $f(x) = x^2 - 5x + 4$, $[1,4]$

cts. on [1,4]? diff. on (1,4)? } yes, polynomials are cts & diff on $(-\infty, \infty)$

$f(1) = 1^2 - 5(1) + 4 = 0$
$f(4) = 4^2 - 5(4) + 4 = 0$) $f(a) = f(b)$

yes, Rolle's Theorem applies

$f'(x) = 2x - 5$

$2x - 5 = 0$

$x = 5/2$

Can the Mean Value Theorem be applied?
If so, find all guaranteed values of c in (a,b).

34. $f(x) = \dfrac{x+1}{x}$, $\left[\frac{1}{2}, 2\right]$

$f'(x) = \dfrac{x \cdot 1 - (x+1)\cdot 1}{x^2} = \dfrac{-1}{x^2}$

<u>Steps to solve MVT problems</u>:
1. Is f continuous on [a,b]? yes
2. Is f differentiable on (a,b)? yes
3. Find (f(b)-f(a))/(b-a)
4. Find f'(x)
5. Set #3&4 equal, solve for x
6. Solution is the values of x from #5 that lie in (a,b)

$$\frac{\frac{2+1}{2} - \frac{\frac{1}{2}+1}{\frac{1}{2}}}{2-\frac{1}{2}} = \frac{\frac{3}{2} - 3}{\frac{3}{2}} = -1$$

$$-\frac{1}{x^2} = -1$$
$$1 = x^2$$
$$\pm 1 = x$$

$x = 1$ $\quad -1 \notin \left(\frac{1}{2}, 2\right)$

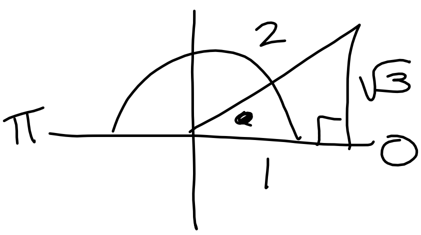
38. $f(x) = 2\sin x + \sin 2x$, $[0, \pi]$

f is continuous on $[0,\pi]$ & differentiable on $(0,\pi)$ } yes, MVT applies

$\dfrac{f(\pi)-f(0)}{\pi - 0} = \dfrac{0}{\pi} = 0$

$f'(x) = 2\cos x + 2\cos 2x$

$$2\cos x + 2[\cos 2x] = 0$$
$$2\cos x + 2[2\cos^2 x - 1] = 0$$
$$\frac{\cancel{2}[\cos x + 2\cos^2 x - 1]}{\cancel{2}} = \frac{0}{2}$$
$$2\cos^2 x + \cos x - 1 = 0$$
$$(2\cos x - 1)(\cos x + 1) = 0$$
$$\cos x = \tfrac{1}{2}, \quad \cos x = -1$$



$x = \dfrac{\pi}{3}$ $\quad \cancel{x = \pi \notin (0,\pi)}$

32. $f(x) = x(x^2 - x - 2) \quad [-1, 1]$

$= x^3 - x^2 - 2x$

f is cts on $[-1,1]$ & diff on $(-1,1) \Rightarrow$ MVT applies

$f'(x) = 3x^2 - 2x - 2$

$$\frac{f(1) - f(-1)}{1-(-1)} = \frac{-2-(-1-1+2)}{2} = -1$$

$$3x^2 - 2x - 2 = -1$$

$$3x^2 - 2x - 1 = 0$$

$$(3x+1)(x-1) = 0$$

$x = -1/3$

$\cancel{x = 1}$