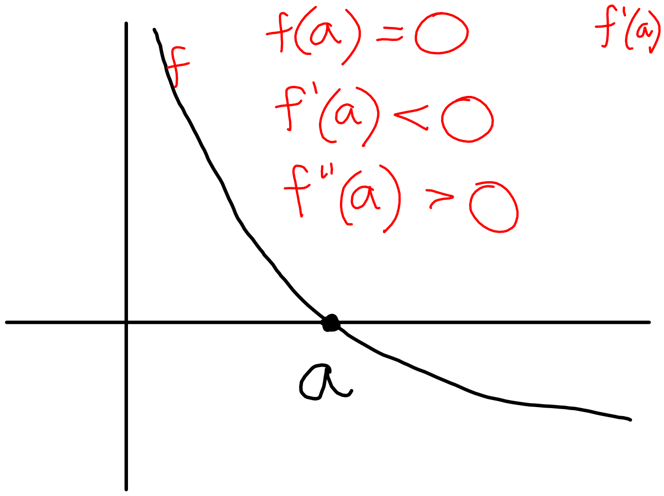
Differential Calculus - Optimization Review & l'Hopital's Rule

October 23, 2017

To find absolute maxima/minima on [a,b],

1. Find critical numbers
2. Plug critical numbers and a and b into original function
3. Whichever f(x) value is largest is absolute max,
whichever f(x) is smallest is absolute min

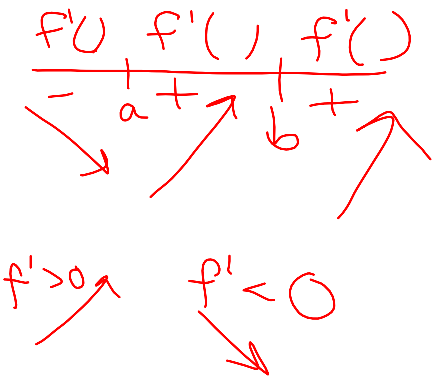


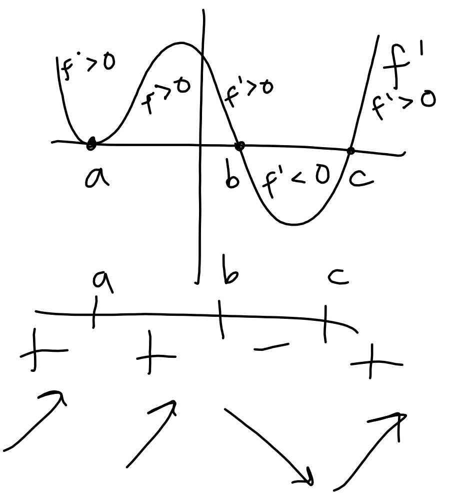
$f(a) = 0$

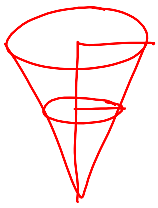
$f'(a) < 0$

$f''(a) > 0$

$f'(a) < f(a) < f''(a)$







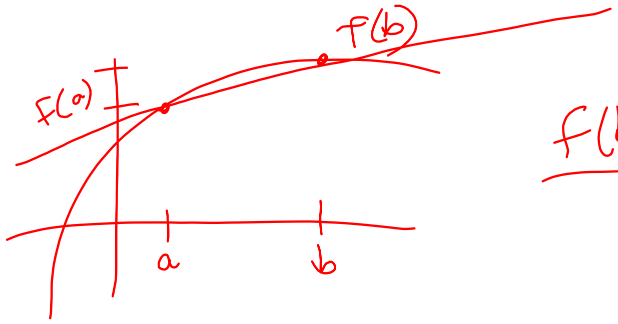
$$V = \frac{1}{3}\pi r^2 h \qquad r = \frac{h}{2}$$

$$= \frac{\pi}{3}\left(\frac{h}{2}\right)^2 h$$

$$V = \frac{\pi}{3}\cdot\frac{h^2}{4}\cdot h$$

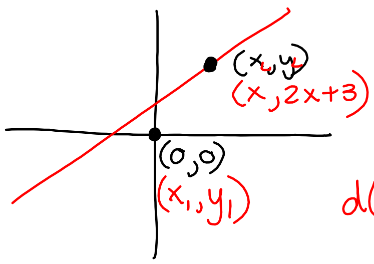
$$V = \frac{\pi}{12}h^3 \Rightarrow \frac{dV}{dt} = \frac{\pi}{12}(3h^2)\cdot\frac{dh}{dt}$$

$$\frac{dV}{dt} = \frac{\pi}{4}h^2\cdot\frac{dh}{dt}$$



$$\frac{f(b)-f(a)}{b-a}$$

What is the x-coordinate of the point on the line $y = 2x + 3$ that is closest to the origin?



1. identify function to optimize
2. rewrite in terms of single variable
3. simplify
4. derive
5. set = 0 & solve

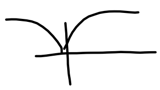
$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

$$d(x)=\sqrt{x^2+\underbrace{(2x+3)^2}_{4x^2+12x+9}}$$

$$d(x)=(5x^2+12x+9)^{1/2}$$

$$d'(x)=\frac{1}{2}(5x^2+12x+9)^{-1/2}\cdot\overbrace{(10x+12)}^{2(5x+6)}$$

$$0=\frac{5x+6}{\sqrt{5x^2+12x+9}}$$



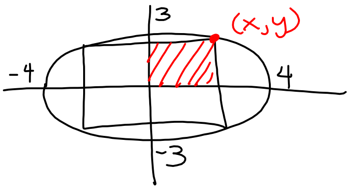
$5x+6=0$

$x=-6/5$

$5x^2+12x+9=0$

$$x=\frac{-12\pm\sqrt{12^2-4(5)(9)}}{2(5)} = \text{not real}$$

What is the area of the largest rectangle with its sides parallel to the x- and y-axes that can be inscribed in the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$?



$$A(x,y)=4xy=\boxed{4(2\sqrt{2})\left(3\sqrt{1-\tfrac{8}{16}}\right)}$$

$$\frac{y^2}{9}=1-\frac{x^2}{16}$$

$$y^2=9\left(1-\frac{x^2}{16}\right)$$

$$y=3\sqrt{1-\frac{x^2}{16}}$$

$$A(x)=4x\cdot 3\sqrt{\frac{16}{16}-\frac{x^2}{16}}=\cancel{4}x\cdot 3\cdot\frac{\sqrt{16-x^2}}{\cancel{\sqrt{16}}\;\cancel{4}}$$

$$A(x)=(3x)(16-x^2)^{1/2}$$

$$A'(x)=3(16-x^2)^{1/2}+(3x)\left[\frac{1}{\cancel{2}}(16-x^2)^{-1/2}(-\cancel{2}x)\right]$$

$$0=3\sqrt{16-x^2}\cdot\frac{\sqrt{16-x^2}}{\sqrt{16-x^2}}-\frac{3x^2}{\sqrt{16-x^2}}$$

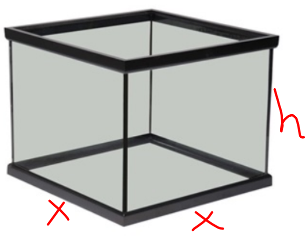
$$0=\frac{3(16-x^2)-3x^2}{\sqrt{16-x^2}}$$

$$0=\frac{-6x^2+48}{\sqrt{16-x^2}}=\frac{-6(x^2-8)}{\sqrt{16-x^2}}$$

$$x^2-8=0$$
$$x^2=8$$
$$x=\pm 2\sqrt{2}$$

An open-topped glass aquarium with a square base is designed to hold 32 cubic feet of water. What is the minimum possible exterior surface area of the aquarium?

[ ] square feet



$$A(x,h)=x^2+4xh\Big|_{\substack{x=4\\h=2}}\quad 4^2+4(4)(2)=48$$

$$32=x^2h$$

$$\frac{32}{x^2}=h$$

$$A(x)=x^2+4x\left(\frac{32}{x^2}\right)$$

$$A(x)=x^2+4(32)x^{-1}$$

$$A'(x)=2x-\frac{4(32)}{x^2}$$

$$\frac{2x^3-4(32)}{x^2}=0$$

$$\cancel{2}x^3=\cancel{4}^{\,2}(32)$$
$$x^3=64\quad x=4$$

7.7 Indeterminate Forms & L'Hôpital's Rule

$\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \cdot \infty$, 1^{∞}, 0^0, *and* $\infty - \infty$ are called indeterminate forms.

L'Hôpital's Rule:

Let f and g be functions that are differentiable on an open interval (a, b) containing c, except possibly at c itself. Assume that $g'(x) \neq 0$ for all x in (a, b), except possibly at c itself. If the limit of f(x)/g(x) as x approaches c produces an indeterminate form $0/0$, ∞/∞, $(-\infty)/\infty$, or $\infty/(-\infty)$, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

7.7

12. $\lim_{x \to -1} \frac{x^2 - x - 2}{x + 1} = \lim_{x \to -1} \frac{(x-2)\cancel{(x+1)}}{\cancel{x+1}} = -1 - 2 = \boxed{-3}$

$= \frac{0}{0} \Rightarrow$ l'hopital's applies

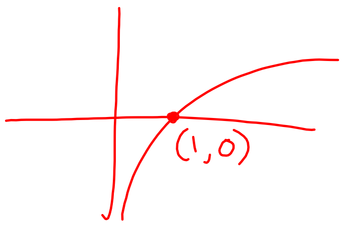
$= \lim_{x \to -1} \frac{2x - 1}{1} = 2(-1) - 1 = \boxed{-3}$

16. $$\lim_{x\to 0^+} \frac{e^x-(1+x)}{x^3} = \frac{e^0-1-0}{0^3} = \frac{0}{0}$$

$$= \lim_{x\to 0^+} \frac{e^x-1}{3x^2} = \frac{0}{0}$$

$$= \lim_{x\to 0^+} \frac{e^x}{6x} = \frac{1}{0^+} = \boxed{\infty}$$

18. $$\lim_{x\to 1} \frac{\ln x^2}{x^2-1} = \frac{0}{0}$$



$$= \lim_{x\to 1} \frac{\frac{1}{x^2}\cdot \cancel{2x}}{\cancel{2x}}$$

$$= \boxed{1}$$

20. $\lim_{x \to 0} \frac{\sin ax}{\sin bx}$ $= \cancel{\lim_{x\to 0} \frac{\frac{\sin ax}{ax}}{\frac{\sin bx}{bx}} \cdot \frac{ax}{bx}}$ $\lim_{x\to 0} \frac{\sin x}{x} = 1$

$= \frac{0}{0}$

$= \lim_{x \to 0} \frac{a \cos ax}{b \cos bx} = \frac{a \cdot 1}{b \cdot 1} = \boxed{\frac{a}{b}}$

$\cos 0 = 1$