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HW due Tues (8th per)/Wed (7th per): 1.2 #1-6 all, 15-22 all, 33,34,39,41

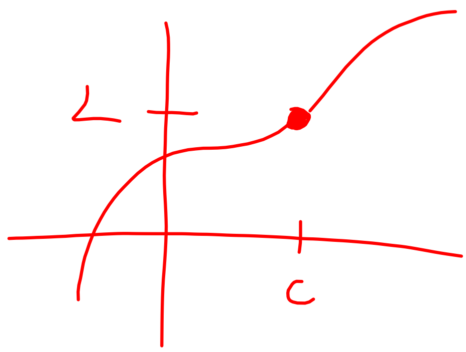
Due Fri: 1.3 #11,21,27-39 odd, 41-61 odd; 83, 87 evaluating limits analytically

- 1.3 #63-73 odd; 89, 90 limits with trig, squeeze theorem
- 1.4 #1-19 odd; limits of functions with discontinuities
- 1.4 #27-30 all; 43-48 all; discuss (dis)continuity
- 1.4 #21,23,25,57,61,65,69,99,102 misc. continuity problems

1.3 Evaluating Limits Analytically

If $\lim\limits_{x \to c} f(x) = f(c)$,

we say that $f(x)$ is continuous at c.



Evaluating Limits Analytically

Basic Limits

Let $b, c \in \mathbb{R}$, $n > 0$ *an integer*, $f, g -$ *functions*, $\lim_{x \to c} f(x) = L$, $\lim_{x \to c} g(x) = K$

1. Constant $\lim_{x \to c} b = b$
2. Identity $\lim_{x \to c} x = c$
3. Polynomial $\lim_{x \to c} x^n = c^n$
4. Scalar Multiple $\lim_{x \to c}[bf(x)] = bL$
5. Sum or Difference $\lim_{x \to c}[f(x) \pm g(x)] = L \pm K$
6. Product $\lim_{x \to c}[f(x)g(x)] = LK$
7. Quotient $\lim_{x \to c}\left[\frac{f(x)}{g(x)}\right] = \frac{L}{K}$, $K \neq 0$
8. Power $\lim_{x \to c}[f(x)]^n = L^n$

$$\lim_{x\to c}[f(x) \pm g(x)] = \lim_{x\to c} f(x) \pm \lim_{x\to c} g(x)$$

$$\lim_{x\to c}[f(x)g(x)] = \left[\lim_{x\to c} f(x)\right]\cdot\left[\lim_{x\to c} g(x)\right]$$

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}, \quad \lim_{x\to c} g(x) \neq 0$$

$$\lim_{x\to c}[f(x)]^n = \left[\lim_{x\to c} f(x)\right]^n$$

Note: If substitution yields $\frac{0}{0}$, an indeterminate form, the expression must be rewritten in order to evaluate the limit.

58. $$\lim_{x\to 0}\frac{\frac{1}{x+4}-\frac{1}{4}}{x} = \lim_{x\to 0}\frac{\frac{1}{x+4}\cdot\frac{4}{4} - \frac{1}{4}\cdot\frac{x+4}{x+4}}{\frac{x}{1}}$$

$$= \lim_{x\to 0}\frac{\frac{4-(x+4)}{4(x+4)}}{\frac{x}{1}} = \lim_{x\to 0}\frac{-\cancel{x}}{4(x+4)}\cdot\frac{1}{\cancel{x}}$$

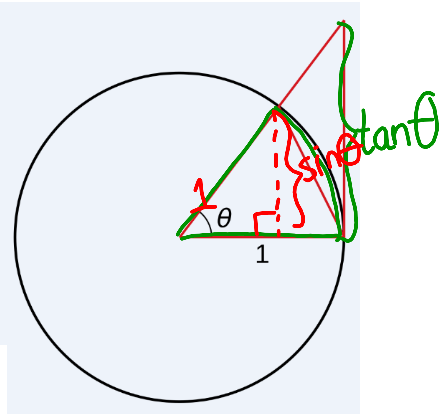
$$= \lim_{x\to 0}\frac{-1}{4(x+4)} = \frac{-1}{4(0+4)} = \boxed{\frac{-1}{16}}$$

66. $\lim_{x\to 2} \frac{x^5-32}{x-2} = \lim_{x\to 2} \frac{\cancel{(x-2)}(x^4+2x^3+4x^2+8x+16)}{\cancel{x-2}}$

$$\begin{array}{r|rrrrrr} 2 & 1 & 0 & 0 & 0 & 0 & -32 \\ & \downarrow & 2 & 4 & 8 & 16 & 32 \\ \hline & 1x^4 & +2x^3 & +4x^2 & +8x & +16 & \boxed{0} \end{array}$$

$\lim_{x\to 2} (x^4+2x^3+4x^2+8x+16) = \boxed{80}$

1.3 The Squeeze Theorem



$\lim_{x\to 0} \frac{\sin x}{x} = ?$

Area of whole circle $= \pi r^2|_{r=1} = \pi$

$\frac{\text{Area of whole circle}}{\text{Total angle of circle}} = \frac{\text{Area of sector}}{\theta}$

$\frac{\pi}{2\pi} = \frac{\text{Area of sector}}{\theta} \rightarrow \text{Area of sector} = \frac{\theta}{2}$

Area of outer triangle $\geq$ Area of sector $\geq$ Area of inner triangle

$$\frac{\tan\theta}{2} \geq \frac{\theta}{2} \geq \frac{\sin\theta}{2}$$

Multiply through by $\frac{2}{\sin\theta}$

$$\frac{\sin\theta}{2\cos\theta}\cdot\frac{2}{\sin\theta} \geq \frac{\theta}{2}\cdot\frac{2}{\sin\theta} \geq \frac{\sin\theta}{2}\cdot\frac{2}{\sin\theta}$$

$$\frac{1}{\cos\theta} \geq \frac{\theta}{\sin\theta} \geq 1$$

Take reciprocals and reverse inequalities

$$\cos\theta \leq \frac{\sin\theta}{\theta} \leq 1$$

Take limits

$$\lim_{\theta\to 0}\cos\theta \quad \le \quad \lim_{\theta\to 0}\frac{\sin\theta}{\theta} \quad \le \quad \lim_{\theta\to 0} 1$$

$$1 \le \lim_{\theta\to 0}\frac{\sin\theta}{\theta} \le 1$$

$$\Rightarrow \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = \boxed{1}$$

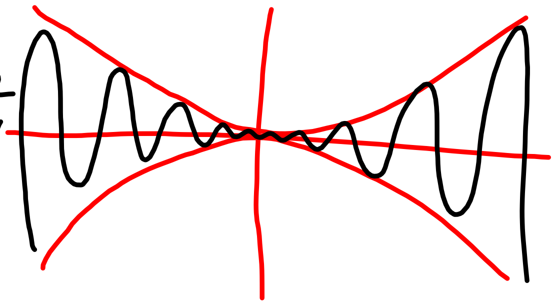
<u>The Squeeze Theorem</u>:

If $f(x) \le g(x) \le h(x)$ *and* $\lim_{x\to c} f(x) = L = \lim_{x\to c} h(x)$,

Then $\lim_{x\to c} g(x) = L$.

$-1 \le \sin x \le 1$ $\quad -x^2 \le x^2 \sin x \le x^2$

$-1 \le \cos x \le 1$



Special Limits Derived by Squeeze Theorem:

$$\boxed{\lim_{x\to 0}\frac{\sin x}{x} = 1} \; ; \; \boxed{\lim_{x\to 0}\frac{1-\cos x}{x} = 0}$$

Memorize!!

Use the squeeze theorem to find

$$\lim_{x\to 0}\left(x^2\cos\frac{5}{x}-3\right)$$

$$-1 \le \cos\frac{5}{x} \le 1$$

$$-x^2 \le x^2\cos\frac{5}{x} \le x^2$$

$$-x^2-3 \le x^2\cos\frac{5}{x}-3 \le x^2-3$$

$$\lim_{x\to 0}\left(-x^2-3\right) \le \lim_{x\to 0}\left(x^2\cos\frac{5}{x}-3\right) \le \lim_{x\to 0}\left(x^2-3\right)$$

$$-3 \le \lim_{x\to 0}\left(x^2\cos\frac{5}{x}-3\right) \le -3$$

By the Squeeze Theorem,

$$\lim_{x\to 0}\left(x^2\cos\frac{5}{x}-3\right) = \boxed{-3}$$

68. $\lim_{x\to 0}\frac{3(1-\cos x)}{x}$

$$\lim_{x\to c} k[f(x)] = k\lim_{x\to c} f(x)$$

$$\lim_{x\to 0}\frac{\sin x}{x} = 1$$

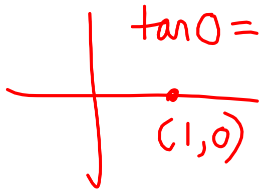
$$\lim_{x\to 0}\frac{1-\cos x}{x} = 0$$

$$= \left(\lim_{x\to 0} 3\right)\left(\lim_{x\to 0}\frac{1-\cos x}{x}\right)$$

$$= 3\cdot\lim_{x\to 0}\frac{1-\cos x}{x}$$

$$= 3\cdot 0 = \boxed{0}$$

72. $\lim\limits_{x\to 0} \dfrac{\tan^2 x}{x} = \lim\limits_{x\to 0} \dfrac{\sin x}{\cos x \cdot \cos x} \cdot \dfrac{\sin x}{1} \cdot \dfrac{1}{x}$



$\tan 0 = \frac{0}{1} = 0$

$= \lim\limits_{x\to 0} \dfrac{\sin x}{x} \cdot \dfrac{\sin x}{\cos^2 x}$

$\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$

$= \left(\lim\limits_{x\to 0} \dfrac{\sin x}{x}\right)\left(\lim\limits_{x\to 0} \dfrac{\sin x}{\cos^2 x}\right)$

$\lim\limits_{x\to 0} \dfrac{1-\cos x}{x} = 0$

$= 1 \cdot \dfrac{\sin 0}{\cos^2 0} = 1 \cdot \dfrac{0}{1} = \boxed{0}$

78. $\lim\limits_{x\to 0} \dfrac{\sin 2x}{\sin 3x}$

$$\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1}$$

$= \lim\limits_{x\to 0} \dfrac{\frac{\sin 2x}{2x} \cdot \frac{2\cancel{x}}{}}{\frac{\sin 3x}{3x} \cdot \frac{3\cancel{x}}{}}$

$\lim\limits_{ax\to 0} \dfrac{\sin ax}{ax} = 1$

If $x \to 0$, $ax \to 0$

$\lim\limits_{x\to 0} \dfrac{\sin ax}{ax} = 1$

$= \dfrac{\lim\limits_{x\to 0} \frac{\sin 2x}{2x}}{\lim\limits_{x\to 0} \frac{\sin 3x}{3x}} \cdot \lim\limits_{x\to 0} \dfrac{2}{3}$

$= \dfrac{1}{1} \cdot \dfrac{2}{3} = \boxed{\dfrac{2}{3}}$