Applications of Derivatives

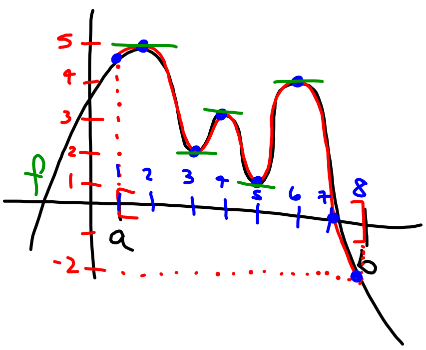
Due: 8th period - Tuesday, 5/2
7th period - Wednesday, 5/3
2.6 # 11-17 odd - Related Rates
Due Friday, 5/5:
2.6 # 19-27 odd, 35 - Related Rates (more challenging problems)

Due Monday 5/8:
3.1 # 17-35 odd - Absolute Extrema on an Interval

Due Tuesday 5/9:
3.2 # 9-21 odd - Rolle's Theorem
3.2 # 33-45 odd - Mean Value Theorem
Due Friday 5/12
3.3 # 17-39 odd - Increasing, Decreasing, and Relative Extrema
3.4 #15-39 odd - Inflection Points and Concavity

3.1 Extrema on an Interval

→ maxima & minima
→ relative & absolute



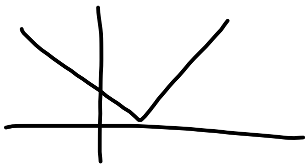
relative minima:
(3,2), (5,1)

relative maxima:
(2,5), (4,3), (6,4)

absolute maximum:
5 @ (2,5)

absolute minimum:
-2 @ (8,-2)

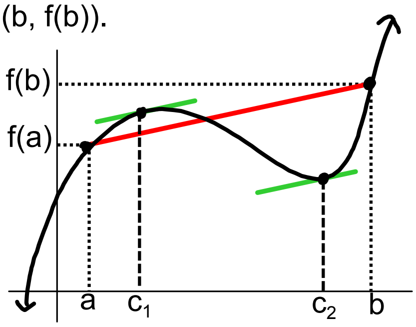
$f(x)$ can have a relative maximum or minimum when $f'(x) = 0$. or $f'(x)$ is undefined.



We call such x-values Critical #'s of f.

3.2 Rolle's Theorem & The Mean Value Theorem

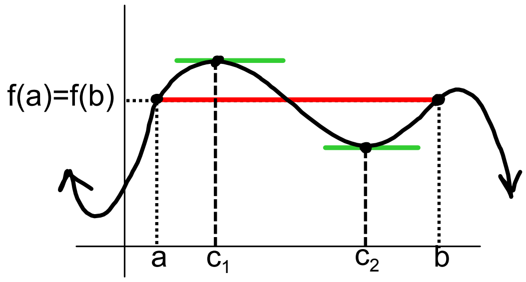
The Mean Value Thereom (MVT) states: If f is continuous on [a,b] and differentiable on (a,b), then there exists at least one c in (a,b) such that the slope of the tangent line at c is equal to the slope of the secant line through (a, f(a)) and (b, f(b)).



If f is continuous on [a,b] and differentiable on (a,b) then $\exists\, c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

Rolle's Theorem is a special case of the MVT where f(a)=f(b),
(and hence involving horizontal secant/tangent lines)



Can the Mean Value Theorem be applied?
If so, find all guaranteed values of c in (a,b).

34. $f(x) = \dfrac{x+1}{x}$, $\left[\frac{1}{2}, 2\right]$

$$\frac{f(2)-f(\frac{1}{2})}{2-\frac{1}{2}} = \frac{\frac{2+1}{2} - \frac{\frac{1}{2}+1}{\frac{1}{2}}}{2-\frac{1}{2}} = \frac{\frac{3}{2}-\frac{3}{2}\cdot\frac{2}{1}}{\frac{3}{2}}$$

$1-2=-1$

f has a non-renovable V.A. @ $x=0 \notin [1/2, 2]$

Steps to solve MVT problems:
1. Is f continuous on [a,b]? } yes,
2. Is f differentiable on (a,b)?
3. Find (f(b)-f(a))/(b-a) $= -1$
4. Find f'(x)
5. Set #3&4 equal, solve for x
6. Solution is the values of x from #5 that lie in (a,b)

$$f'(x) = \frac{x(1)-(x+1)\cdot 1}{x^2}$$

$$\frac{d}{dx}\left(\frac{x+1}{x}\right) = \frac{-1}{x^2}$$

$$-\frac{1}{x^2} = -1 \qquad -1 = -x^2 \qquad 1 = x^2 \qquad \cancel{\pm}1 = x \qquad \boxed{x=1}$$

$$\text{solve}\left(\frac{-1}{x^2} = -1, x\right)$$

38. $f(x) = 2\sin x + \sin 2x$, $[0, \pi]$

yes, f is continuous on $[0,\pi]$ & differentiable on $(0,\pi)$ so MVT applies

$$\frac{f(\pi)-f(0)}{\pi - 0} = \frac{2\sin(\pi) + \sin(2\pi) - (2\sin(0) + \sin(2\cdot 0))}{\pi - 0} = 0$$

$$f'(x) = \frac{d}{dx}\left(2\sin(x) + \sin(2x)\right) = 2\cos x + 2\cos 2x$$

solve$\left(2\cos(x) + 2\cos(2x) = 0, x\right)$

$2\cos x + 2(2\cos^2 x - 1) = 0$

$2\cos x + 4\cos^2 x - 2 = 0$

$2(2\cos^2 x + \cos x - 1) = 0$

$2(2\cos x - 1)(\cos x + 1) = 0$

$\cos x = \frac{1}{2}$ $\quad$ $\cos x = -1$

$\boxed{x = \pi/3}$ $\quad$ $\cancel{x = \pi} \notin (0, \pi)$

$x = -\frac{\pi}{3} + 2\pi k = \frac{(6n-1)\pi}{3}$,

$\frac{\pi}{3} + 2\pi k = \frac{(6n+1)\pi}{3}$,

$-\pi + 2\pi k = (2n-1)\pi$

32. $f(x) = x(x^2 - x - 2)$ $[-1, 1]$

polynomials are continuous & differentiable everywhere; Hence yes, MVT applies

$$\frac{f(1)-f(-1)}{1-(-1)} = \frac{1\cdot(1-1-2) - (-1)\cdot((-1)^2 - (-1) - 2)}{1-(-1)} = -1$$

$$f'(x) = \frac{d}{dx}\left(x\cdot(x^2 - x - 2)\right) = (x^3 - x^2 - 2x)' = 3x^2 - 2x - 2$$

Solve$\left(3x^2 - 2x - 2 = -1, x\right)$ $\quad$ $\boxed{x = \frac{-1}{3}}$

$3x^2 - 2x - 1 = 0$

$(3x+1)(x-1) = 0$ $\quad$ $\cancel{x = 1} \notin (-1, 1)$

3.3-3.4 Increasing, Decreasing, Concavity, and the 1st and 2nd Derivative Tests

What do *f′* and *f″* tell us about *f*?

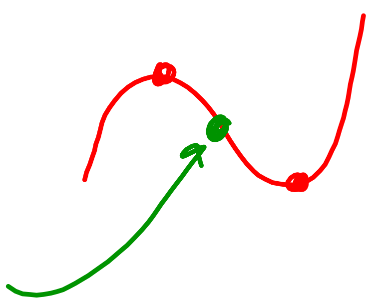
Recall that *f′* is the rate of change or slope of *f*,
f″ is the slope or rate of change of *f′*.

f'	f
+	↗ increasing
−	↘ decreasing

f''	f'	f
+	↗ increasing	concave up
−	↘ decreasing	concave down

f′(x)=0 when f has a relative maximum or minimum.
These x-values (and those where f′(x) is undefined) are called <u>critical numbers</u>.

f″(x)=0 when f changes concavity.
The points where concavity changes are called <u>inflection points</u>.



To solve problems involving concavity, increasing/decreasing, etc., we should recall how to solve polynomial inequalities.

$-4 < x \le -2$ or $x \ge 3$

$$\frac{(x+2)(x-3)}{x+4} \ge 0$$

$$\text{solve}\left(\frac{(x+2)\cdot(x-3)}{x+4} \ge 0,\ x\right)$$

$f'(-5)$ $f'(-3)$ $f'(0)$ $f'(4)$

$-$ -4 $+$ -2 $-$ 3 $+$

f'

f''

- Find all critical numbers and state the open intervals on which f is increasing and/or decreasing.
- Find all inflection points and state the open intervals on which f is concave up and/or concave down.
- Use these results to determine all relative and absolute extrema.

3.3

16. $f(x) = x^3 - 6x^2 + 15$

testing to see where f is incr/decr & looking for extrema

$f'(x) = 3x^2 - 12x = 3x(x-4) = 0$

critical #'s: $0, 4$

$f'(-1)$, $f'(1)$, $f'(5)$

$+$ 0 $-$ 4 $+$

f is increasing on $(-\infty, 0) \cup (4, \infty)$

f is decreasing on $(0, 4)$

f has a relative maximum @ $(0, f(0)) = (0, 15)$

f has a relative minimum @ $(4, f(4)) = (4, -17)$

$f''(x) = 6x - 12 = 6(x-2) = 0$

$x = 2$

testing for concavity & inflection pts.

$f''(0)$, $f''(3)$

$-$ 2 $+$

f is concave down on $(-\infty, 2)$

f is concave up on $(2, \infty)$

f has an inflection point @ $(2, f(2)) = (2, -1)$