4.4 – The Fundamental Theorem of Calculus

$F'(x) = f(x)$

<u>1st Fundamental Theorem of Calculus:</u>
If a function f is continuous on the closed interval $[a,b]$ and F is an antiderivative of f on the interval $[a,b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

$F'(c) = \frac{F(b)-F(a)}{b-a}$

$(b-a)F'(c) = F(b) - F(a)$

$(b-a)f(c) = F(b) - F(a)$

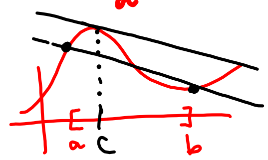
$(b-a)f(c) = \int_a^b f(x)\,dx$

<u>Mean Value Theorem for Integrals:</u>
If a function f is continuous on the closed interval $[a,b]$, then there exists a number c in the closed interval $[a,b]$ such that

$$\int_a^b f(x)\,dx = f(c)(b-a)$$

Recall the <u>Mean Value Theorem:</u>
If f is continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b), then there exists a number c in (a,b) such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$



Find the value(s) of c guaranteed by the Mean Value Theorem for Integrals for the function over the indicated interval.

$\int_a^b f(x)\,dx = f(c)\cdot(b-a)$

46. $f(x) = \frac{9}{x^3}, [1,3]$

MVT-I applies

$\int_1^3 \frac{9}{x^3}\,dx = \frac{9}{c^3}(3-1)$

$-\frac{9}{2}x^{-2}\Big|_1^3 = \frac{9}{c^3}(2)$

$\frac{-9}{2(3)^2} - \left(\frac{-9}{2(1)^2}\right) = \frac{18}{c^3}$

$\frac{-1}{2} + \frac{9}{2} = \frac{18}{c^3}$

$4c^3 = 18$

$c^3 = \frac{18}{4} = \frac{9}{2}$

$\boxed{c = \sqrt[3]{9/2}}$

Average value of a function on an interval:

If f is integrable on the closed interval $[a, b]$, then the average value of f on the interval is

MVT-I:

$$\int_a^b f(x)dx = f(c)\cdot(b-a)$$

$$\frac{1}{b-a}\int_a^b f(x)dx = f(c)$$

$$\frac{1}{b-a}\int_a^b f(x)\,dx = f(c)$$

avg. value of the function over $[a, b]$

50. $f(x) = \dfrac{4(x^2+1)}{x^2}$, $[1, 3]$

avg. value: $\dfrac{1}{3-1}\displaystyle\int_1^3 (4 + 4x^{-2})\,dx$

$$= \frac{1}{2}\cdot\left[4x - 4x^{-1}\right]_{x=1}^{3} = \frac{1}{2}\left(4\cdot 3 - \frac{4}{3} - \left(4\cdot 1 - \frac{4}{1}\right)\right)$$

$$= 6 - \frac{2}{3} = \boxed{\frac{16}{3}}$$

The Second Fundamental Theorem of Calculus:

If f is continuous on an open interval I containing a, then, for every x in the interval,

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$$

"Fix x"

76. $F(x) = \int_0^x t(t^2+1)\,dt = \int_0^x (t^3+t)\,dt$ Find $F'(x)$

rewrite $F(x)$ as a function of x.

$$\left[\frac{t^4}{4}+\frac{t^2}{2}\right]_0^x = \frac{x^4}{4}+\frac{x^2}{2} - \left(\frac{0^4}{4}+\frac{0^2}{2}\right)$$

Verify using 2nd FTC.

$$\int_0^x t(t^2+1)\,dt = \frac{x^4}{4}+\frac{x^2}{2}$$

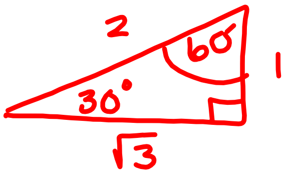
$$\frac{d}{dx}\int_0^x t(t^2+1)\,dt = \frac{d}{dx}\left(\frac{x^4}{4}+\frac{x^2}{2}\right)$$

$$x(x^2+1) = x^3+x \checkmark$$

80. $\displaystyle\int_{\pi/3}^x \sec t\tan t\,dt = \sec t\Big|_{\pi/3}^x$

$= \sec x - \sec \pi/3$

$= \sec x - 2$



$$\frac{d}{dx}\int_{\pi/3}^x \sec t\tan t\,dt = \frac{d}{dx}(\sec x - 2) \checkmark$$

86. $F(x) = \int_0^x \sec^3 t \, dt$ Find $F'(x)$

$F'(x) = \boxed{\sec^3 x}$

What about $F'(x)$ when $F(x) = \int_a^{g(x)} f(t)\,dt$?

Let $g(x) = u$

$F'(x) = \frac{dF}{dx} = \frac{dF}{du} \cdot \frac{du}{dx} = \frac{d}{du}\left[F\right] \cdot \frac{du}{dx}$

$= \frac{d}{du}\left[\int_a^u f(t)\,dt\right] \cdot \frac{du}{dx} = f(u) \cdot \frac{du}{dx}$

(i.e. we have chain rule)

90. $F(x) = \int_2^{x^2} \frac{1}{t^3}\,dt = \int_2^{u} \frac{1}{t^3}\,dt$

Let $u = x^2$

$\frac{du}{dx} = 2x$

$F'(x) = \frac{dF}{dx} = \frac{dF}{du}\cdot\frac{du}{dx} = \frac{d}{du}\left[F(x)\right]\cdot\frac{du}{dx}$

$= \frac{d}{du}\left[\int_2^{u}\frac{1}{t^3}\,dt\right]\cdot\frac{du}{dx}$

$= \frac{1}{u^3}\cdot\frac{du}{dx} = \frac{1}{(x^2)^3}\cdot 2x = \frac{2x}{x^6} = \boxed{\frac{2}{x^5}}$

92. $F(x) = \int_0^{x^2} \sin\theta^2\,d\theta = \cancel{\int_0^{u} \sin\theta^2\,d\theta}$

$\cancel{u = x^2}$

$F'(x) = \boxed{\sin(x^2)^2 \cdot 2x}$

HW
4.4 : 45-51 odd
75-91 odd

4.5 # 7-33, 41-53,
57-63 odd