3.9 - Differentials

Recall:

For a function f that is differentiable at c, the equation of the tangent line at the point $(c, f(c))$ is given by

$y - f(c) = f'(c)(x - c)$

This follows from the point-slope equation $y - y_1 = m(x - x_1)$, where the slope m is the derivative $f'(x)$ evaluated at the point $(c, f(c))$.

Since c, $f(c)$, and $f'(c)$ are all constants, if we rearrange to solve for y,

$y = f(c) + f'(c)(x - c)$

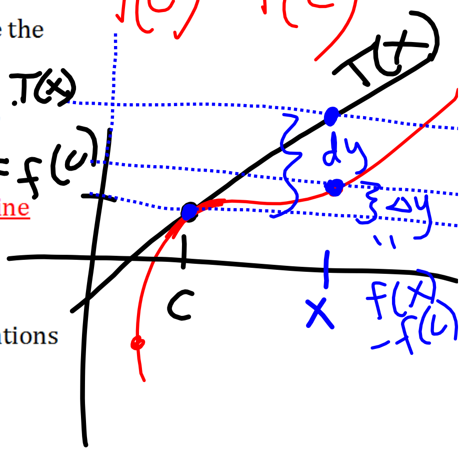
y is a linear function of x, called the linear approximation or tangent line approximation to the graph of $f(x)$ at $x = c$.

$T(x) = f(c) + f'(c)(x - c)$

For values of x close to c, values of $y = T(x)$ can be used as approximations of the values of the original function f.

$T(c) = f(c) + f'(c) \cdot 0$

$T(c) = f(c)$



Recall that the slope of the *secant line* through two points $(c, f(c))$ and $(x, f(x))$ is given by $\frac{\Delta y}{\Delta x} = \frac{f(x)-f(c)}{x-c}$, and the slope of the *tangent line* is the limit as the distance between these two points goes to zero of this expression, which we define to be the derivative.

$m = \frac{\Delta y}{\Delta x}$

Noting that the change in x is $\Delta x = x - c$, or $x = c + \Delta x$ and hence $f(x) = f(c + \Delta x)$, we can write this two ways:

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x}$$

Actual change in y is $\Delta y = f(x) - f(c) = f(c + \Delta x) - f(c)$.

$\Delta y = y_2 - y_1$
$= f(x) - f(c)$
$= f(c + \Delta x) - f(c)$

Recalling the tangent line *approximation* equation

$\boxed{T(x) = f(c) + f'(c)(x - c)} = f(c) + f'(c)\Delta x$

$T(x) - f(c) = f'(c)\Delta x$

We can see that change in y can be approximated by $T(x) - f(c)$, or

Approximate change in y is $\Delta y \approx f'(c)\Delta x$.

For such an approximation, Δx is denoted dx, and is called the differential of x. The expression $f'(x)dx$ is denoted by dy and called the **differential of y**.

$$\boldsymbol{dy = f'(x)dx}$$

In many applications, the differential of y can be used as an approximation of the actual change in y, i.e. $\Delta y \approx f'(x)dx$

All of the differentiation rules can be written in differential form.

By definition of differentials, we have for functions (of x) u and v:

$du = u'dx$ and $dv = v'dx$

Note that rearranged, these look like $\frac{du}{dx} = u'$ and $\frac{dv}{dx} = v'$.

For example, the Product Rule becomes:

$d[uv] = [uv]'dx = [uv' + vu']dx = uv'dx + vu'dx = udv + vdu$

$$d(y) = f'(x) \cdot dx$$

$$\frac{dy}{dx} = f'(x)$$

Differential Formulas

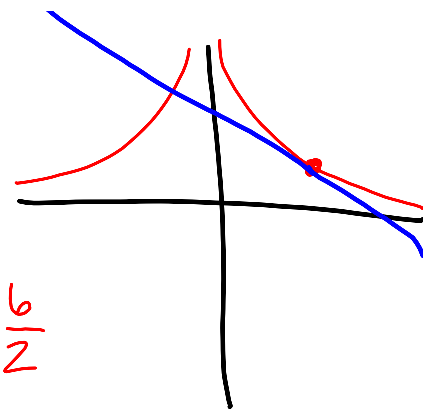
Constant multiple: $d[cu] = cdu$

Sum or difference: $d[u \pm v] = du \pm dv$

Product: $d[uv] = udv + vdu$

Quotient: $d\left[\frac{u}{v}\right] = \frac{vdu - udv}{v^2}$

3.9 #2 $f(x) = \frac{6}{x^2}$; $\left(2, \frac{3}{2}\right) = (c, f(c))$



Compare the actual function values with the tangent line approximation near 2.

$f(x) = 6x^{-2}$

$f'(x) = -12x^{-3} = \frac{-12}{x^3}$

$f'(2) = \frac{-12}{2^3} = \frac{-12}{8} = \frac{-3}{2}$

Tangent line $T(x)$: $y = f(c) + f'(c)(x - c)$

$$T(x) = \frac{3}{2} + \frac{-3}{2}(x - 2) = \frac{3}{2} - \frac{3}{2}x + \frac{6}{2}$$

$$T(x) = -\frac{3}{2}x + \frac{9}{2}$$

x	1.9	1.99	2	2.01	2.1	2.5
$f(x)$	1.662	1.515	1.5	1.485	1.360	0.960
$T(x)$	1.650	1.515	1.5	1.485	1.350	0.750