4.4 – The Fundamental Theorem of Calculus

1st Fundamental Theorem of Calculus:

If a function f is continuous on the closed interval $[a, b]$ and F is an antiderivative of f on the interval $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Mean Value Theorem for Integrals:

If a function f is continuous on the closed interval $[a, b]$, then there exists a number c in the closed interval $[a, b]$ such that

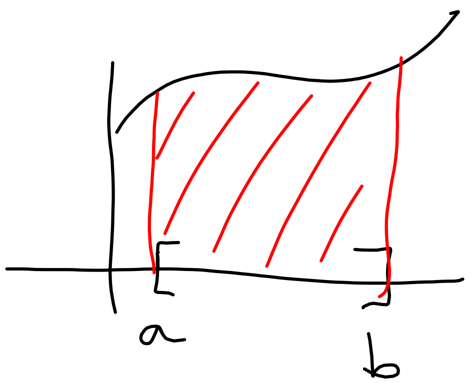
$$\int_a^b f(x)\,dx = f(c)(b-a) \rightarrow$$

the average value of f on $[a,b]$ is $\frac{1}{b-a}\int_a^b f(x)\,dx$.

Recall the Mean Value Theorem:

If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



$$\int_a^b f(x)\,dx = \text{area}$$

width $= b - a$

height $= f(x)$ avg

$$\text{avg. } f(x) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Find the value(s) of c guaranteed by the Mean Value Theorem for Integrals for the function over the indicated interval.

46. $f(x) = \frac{9}{x^3}, [1,3]$

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$(b-a)\cdot f(c) = \int_a^b f(x)\,dx$$

$$\frac{9}{c^3} = \frac{1}{3-1}\int_1^3 \frac{9}{x^3}\,dx$$

$$\frac{9}{c^3} = \frac{1}{2}\cdot\int_1^3 9x^{-3}\,dx$$

$$\frac{9}{c^3} = \frac{1}{2}\cdot\left(-\frac{9}{2x^2}\Big|_1^3\right)$$

$$\frac{9}{c^3} = \frac{1}{2}\left(\frac{-9}{2\cdot 3^2} - \frac{-9}{2\cdot 1^2}\right)$$

$$\frac{9}{c^3} = \frac{1}{2}\left(-\frac{1}{2} + \frac{9}{2}\right)$$

$$9 = 2c^3$$

$$\frac{9}{2} = c^3$$

$$\boxed{c = \sqrt[3]{\frac{9}{2}}}$$

Average value of a function on an interval:

If f is integrable on the closed interval $[a,b]$, then the average value of f on the interval is

$$\frac{1}{b-a}\int_a^b f(x)\,dx$$

50. $f(x) = \frac{4(x^2+1)}{x^2}$, $[1,3]$

$$\int 4x^{-2}\,dx = \frac{4x^{-2+1}}{-2+1} = \frac{4x^{-1}}{-1} = -\frac{4}{x}$$

avg. value: $\frac{1}{3-1}\int_1^3 (4+4x^{-2})\,dx$

$$\frac{4x^2+4}{x^2} = \frac{4x^2}{x^2} + \frac{4}{x^2} = 4+4x^{-2}$$

$$\frac{1}{2}\cdot\left(4x - \frac{4}{x}\right)\Big|_1^3 = 2x - \frac{2}{x}\Big|_1^3 = \left(6-\frac{2}{3}\right) - (2-2) = \boxed{\frac{16}{3}}$$

The Second Fundamental Theorem of Calculus:

If f is continuous on an open interval I containing a, then, for every x in the interval,

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$$

"Fix x"

76. $F(x) = \int_0^x t(t^2+1)\,dt = \int_0^x (t^3+t)\,dt$

rewrite $F(x)$ as a function of x.

$$\left[\frac{t^4}{4} + \frac{t^2}{2}\right]_0^x = \frac{x^4}{4} + \frac{x^2}{2} - \left(\frac{0^4}{4} + \frac{0^2}{2}\right)$$

verify using 2nd FTC.

$$\int_0^x t(t^2+1)\,dt = \frac{x^4}{4} + \frac{x^2}{2}$$

$$\frac{d}{dx}\int_0^x t(t^2+1)\,dt = \frac{d}{dx}\left(\frac{x^4}{4} + \frac{x^2}{2}\right)$$

$$x(x^2+1) = x^3+x \checkmark$$

Rewrite F as a function of x.

80. $\int_{\pi/3}^{x} \sec t \tan t \, dt = \sec t \Big|_{\pi/3}^{x}$

Find F'(x)

86. $F(x) = \int_{0}^{x} \sec^3 t \, dt$

$F'(x) = \boxed{\sec^3 x}$

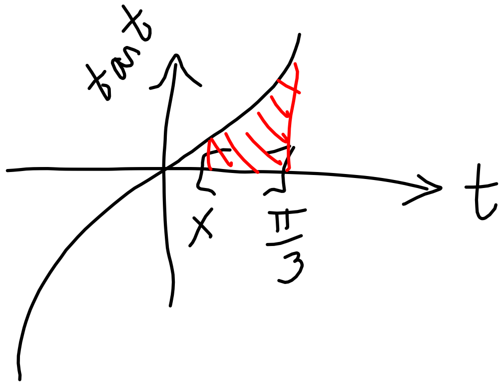
2nd FTC

If $F(x) = \int_{a}^{x} f(t)\,dt$,

then $F'(x) = f(x)$

$$F(x) = \int_x^{\pi/3} \tan t \, dt = -\int_{\pi/3}^{x} \tan t \, dt$$

$$F'(x) = \boxed{-\tan x}$$



What about $F'(x)$ when $F(x) = \int_a^{g(x)} f(t)\,dt$?

Let $g(x) = u$

$$F'(x) = \frac{dF}{dx} = \frac{dF}{du} \cdot \frac{du}{dx} = \frac{d}{du}\left[F\right] \cdot \frac{du}{dx}$$

$$= \frac{d}{du}\left[\int_a^u f(t)\,dt\right] \cdot \frac{du}{dx} = f(u) \cdot \frac{du}{dx}$$

(i.e. we have chain rule)

If $F(x) = \int_a^{g(x)} f(t)\,dt$

then $F'(x) = f(g(x)) \cdot g'(x)$

90. $F(x) = \int_2^{x^2} \frac{1}{t^3}\,dt = \int_2^{u} \frac{1}{t^3}\,dt$

Let $u = x^2$

$\frac{du}{dx} = 2x$

$F'(x) = \frac{dF}{dx} = \frac{dF}{du} \cdot \frac{du}{dx} = \frac{d}{du}\left[F(x)\right] \cdot \frac{du}{dx}$

$= \frac{d}{du}\left[\int_2^{u} \frac{1}{t^3}\,dt\right] \cdot \frac{du}{dx}$

$= \frac{1}{u^3} \cdot \frac{du}{dx} = \frac{1}{(x^2)^3} \cdot 2x = \frac{2x}{x^6} = \boxed{\frac{2}{x^5}}$

$F'(x) = \frac{1}{(x^2)^3} \cdot 2x$

$= \frac{1}{x^6} \cdot 2x$

$= \boxed{\frac{2}{x^5}}$

92. $F(x) = \int_0^{x^2} \sin\theta^2 \, d\theta = \int_0^{u} \sin\theta^2 \, d\theta$

$u = x^2$

$$F'(x) = \sin(u^2) \cdot \frac{du}{dx}$$

$$= \sin\left(x^2\right)^2 \cdot 2x$$

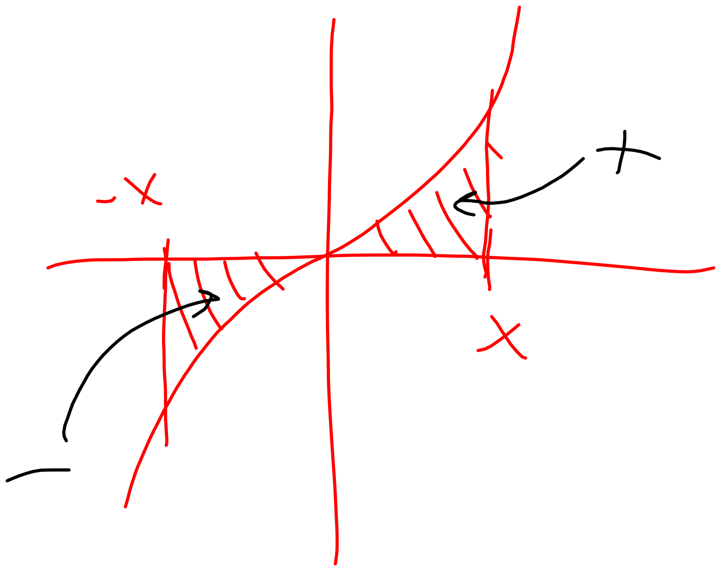
$$= \boxed{2x\sin(x^4)}$$

88. $F(x) = \int_{-x}^{x} t^3 \, dt = \int_{-x}^{a} t^3 \, dt + \int_{a}^{x} t^3 \, dt$

$$F(x) = -\int_a^{-x} t^3 \, dt + \int_a^{x} t^3 \, dt$$

$$F'(x) = -(-x)^3 \underbrace{(-1)}_{(-x)'} + x^3$$

$$= \boxed{0}$$



$$\boxed{\frac{d}{dx}\int_a^{g(x)} f(t)\,dt = f(g(x))\cdot g'(x)}$$

4.4 #45-55 odd MVT for integrals & avg fn value

#81-92 all 2nd FTC

Find $F'(x)$.

81. $F(x)=\int_{-2}^{x}(t^2-2t)\,dt$

82. $F(x)=\int_{1}^{x}\frac{t^2}{t^2+1}\,dt$

83. $F(x)=\int_{-1}^{x}\sqrt{t^4+1}\,dt$

84. $F(x)=\int_{1}^{x}\sqrt[4]{t}\,dt$

85. $F(x)=\int_{0}^{x}t\cos t\,dt$

86. $F(x)=\int_{0}^{x}\sec^3 t\,dt$

87. $F(x)=\int_{x}^{x+2}(4t+1)\,dt$

88. $F(x)=\int_{-x}^{x}t^3\,dt$

89. $F(x)=\int_{0}^{\sin x}\sqrt{t}\,dt$

90. $F(x)=\int_{2}^{x^2}\frac{1}{t^3}\,dt$

91. $F(x)=\int_{0}^{x^3}\sin t^2\,dt$

92. $F(x)=\int_{0}^{x^2}\sin\theta^2\,d\theta$

$$F(x)=\int_{g(x)}^{h(x)}f(t)\,dt=\int_{g(x)}^{a}f(t)\,dt+\int_{a}^{h(x)}f(t)\,dt$$

$$=-\int_{a}^{g(x)}f(t)\,dt+\int_{a}^{h(x)}f(t)\,dt$$