Fractals, cont.

1 segment of length 1

$\ell_0 = 1$

4 seg. ea. $\ell = \frac{1}{3}$

$\ell_1 = \frac{4}{3}$

16 seg. of $\ell = \frac{1}{9}$

$\ell_2 = \frac{16}{9} = \left(\frac{4}{3}\right)^2$

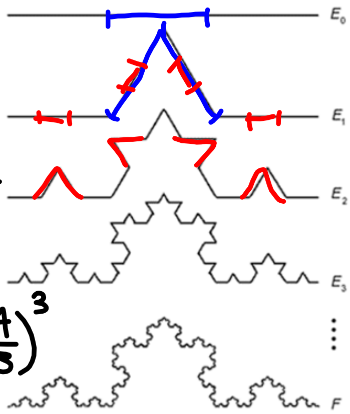
64 seg of $\ell = \frac{1}{27}$

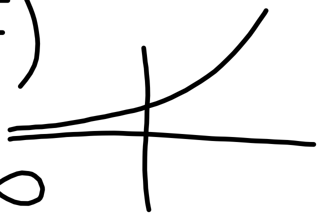
$\ell_3 = \frac{64}{27} = \left(\frac{4}{3}\right)^3$

$\ell_n = \left(\frac{4}{3}\right)^n$

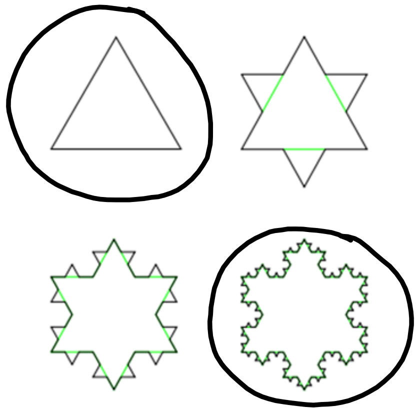
$\ell_k = \infty$

Koch Curve:





Koch Snowflake:

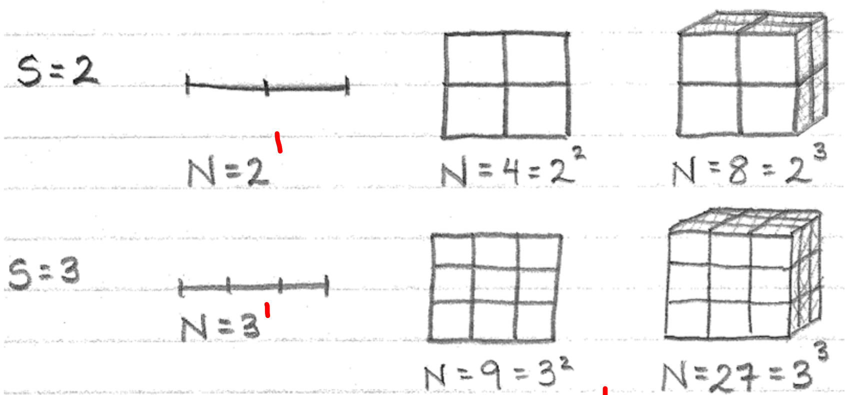


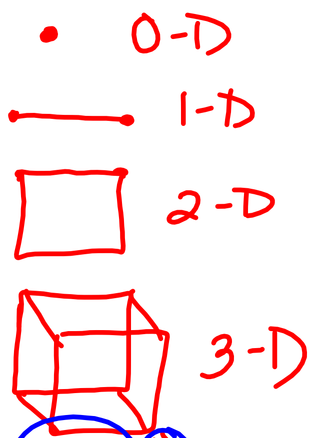
curve of infinite perimeter enclosing a finite area

Fractal Dimension:

Recall that points in space are 0-dimensional; lines are 1-dimensional; a square is 2-dimensional; and a cube is 3-dimensional. Fractals don't behave exactly like objects in these integer dimensions.

Suppose that an object has the following property: if we scale it down by a factor of **S**, then the object can be built from **N** scaled-down versions of itself.





$N = S^d$, where $d =$ dimension

$d = \frac{\log N}{\log S} =$ fractal dimension

$N = S^d$

$\ln(N) = \ln(S^d)$

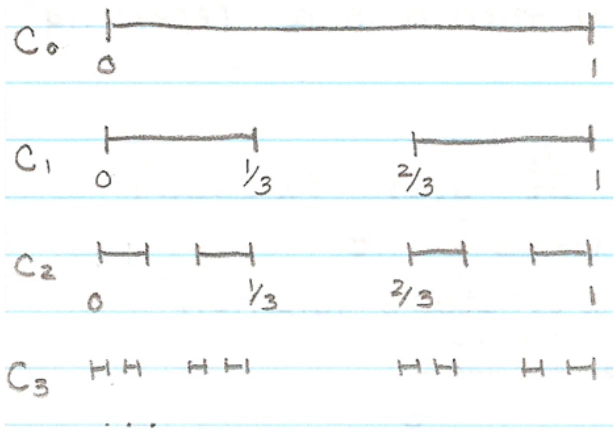
$\ln N = d \cdot \ln S$

$\boxed{d = \frac{\ln N}{\ln S}}$

The Cantor Middle-Third Set:

scaling factor

$S = 3$



$N = 2$

It takes 2 shrunk-down versions of C_{n-1} to make up C_n

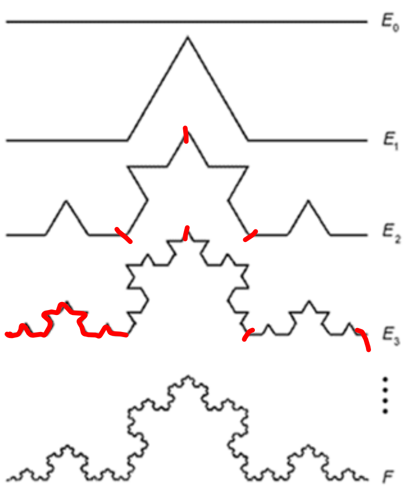
Suppose that an object has the following property: if we scale it down by a factor of **S**, then the object can be built from **N** scaled-down versions of itself.

$d = \frac{\log N}{\log S}$ = <u>fractal dimension</u>

$$d = \frac{\ln N}{\ln S} = \frac{\ln 2}{\ln 3} = 0.63$$

<u>Ex</u> <u>Cantor Set</u> $S = 3$, $N = 2$, $d = \frac{\log 2}{\log 3} \approx 0.63$

Koch Curve:



$S = 3$

$N = 4$

$$d = \frac{\ln N}{\ln S} = \frac{\ln 4}{\ln 3} \approx 1.62$$

$S = 3$, $N = 4$, $d = \frac{\log 4}{\log 3} \approx 1.62$

Sierpinski Triangle:

T_0 T_1 T_2 T_3 T_4

Let T_n be the nth iteration of the Sierpinski triangle;
let A_n be the area of the nth iteration of the Sierpinski triangle; let $A_0 = 1$;
let P_n be the perimeter of the nth iteration of the Sierpinski triangle.

T_n consists of 3^n triangles of area $\left(\frac{1}{4}\right)^n$;

How much was removed?

We remove 3^n triangles of area $\left(\frac{1}{4}\right)^{n+1}$ to get T_{n+1} from T_n.

Hence the total area removed is:

$$\sum_{n=0}^{\infty} \frac{1}{4}\left(\frac{3}{4}\right)^n = \frac{\frac{1}{4}}{1-\frac{3}{4}} = \frac{\frac{1}{4}}{\frac{1}{4}} = 1$$

$P_{n+1} = \frac{3}{2} P_n$, which implies that $P_n = \left(\frac{3}{2}\right)^n$,

which approaches ∞ as n approaches ∞.

The Sierpinski triangle has zero area but an infinite perimeter!

Fractal Dimension: $S = 2$, $N = 3$, $d = \frac{\log 3}{\log 2} \approx 1.58$

	T_0	T_1
	T_0	T_1
~~Perimeter of this iteration~~	~~$P_0 = \frac{6}{\sqrt[4]{3}}$~~	~~$P_1 = \frac{9}{\sqrt[4]{3}}$~~
~~Perimeter of this iteration divided by perimeter of previous iteration~~	~~N/A~~	~~$\frac{P_1}{P_0} = \frac{3}{2}$~~
Area removed from previous iteration to obtain this iteration	N/A	$1/4$
Total area removed up to this point	0	$1/4$
Number of triangles in this iteration	1	3
Area of a triangle in this iteration	1	$1/4$
Total area of this iteration	$A_0 = 1$	$A_1 = 3/4$

T_0 T_1 T_2 T_3 T_4 T_n

	T_0	T_1	T_2	T_3	T_4	T_n
~~Perimeter of this iteration~~	~~$P_0 = \frac{6}{\sqrt[4]{3}}$~~	~~$P_1 = \frac{9}{\sqrt[4]{3}}$~~	~~$P_2 =$~~	~~$P_3 =$~~	~~$P_4 =$~~	
~~Perimeter of this iteration divided by perimeter of previous iteration~~	~~N/A~~	~~$\frac{P_1}{P_0} = \frac{3}{2}$~~	~~$\frac{P_2}{P_1} =$~~	~~$\frac{P_3}{P_2} =$~~	~~$\frac{P_4}{P_3} =$~~	
Area removed from previous iteration to obtain this iteration	N/A	$\frac{1}{4} = \frac{3^0}{4^1}$	$\frac{3}{16} = \frac{3^1}{4^2}$	$\frac{9}{64} = \frac{3^2}{4^3}$	$3^3/4^4$	$\frac{3^{n-1}}{4^n}$
Total area removed up to this point	0	$1/4$	$\frac{3^0}{4^1} + \frac{3^1}{4^2}$	$\frac{3^0}{4^1} + \frac{3^1}{4^2} + \frac{3^2}{4^3}$	$\sum_{i=0}^{3} \frac{3^i}{4^{i+1}}$	$\sum_{i=0}^{n-1} \frac{3^i}{4^{i+1}}$
Number of triangles in this iteration	1	3	$9 = 3^2$	$27 = 3^3$	$81 = 3^4$	3^n
Area of a triangle in this iteration	1	$1/4$	$\frac{1}{16} = \frac{1}{4^2}$	$\left(\frac{1}{4}\right)^3$	$\left(\frac{1}{4}\right)^4$	$\left(\frac{1}{4}\right)^n$
Total area of this iteration	$A_0 = 1$	$A_1 = 3/4$	$A_2 = \frac{9}{16} = \left(\frac{3}{4}\right)^2$	$A_3 = \left(\frac{3}{4}\right)^3$	$A_4 = \left(\frac{3}{4}\right)^4$	$\left(\frac{3}{4}\right)^n$
Area of this iteration divided by area of previous iteration	N/A	$\frac{A_1}{A_0} = 3/4$	$\frac{A_2}{A_1} = 3/4$	$\frac{A_3}{A_2} = 3/4$	$\frac{A_4}{A_3} = 3/4$	

The initial triangle T_0 had area $A_0 = 1$.

Total area removed up to the n^{th} iteration is $\sum_{i=0}^{n-1} \frac{3^i}{4^{i+1}}$

$$= \sum_{i=1}^{n} \frac{3^{i-1}}{4^i}$$

$\frac{3^{1-1}}{4^1} = \frac{3^0}{4} = \frac{1}{4}$ $\quad \frac{3^{n-1}}{4^n}$

$\frac{3^0}{4^{0+1}} = \frac{1}{4}$ $\quad \frac{3^{n-1}}{4^{n-1+1}} = \frac{3^{n-1}}{4^n}$

$$S_\infty = \frac{a_1}{1-r} = \frac{\frac{1}{4}}{1-\frac{3}{4}} = \frac{\frac{1}{4}}{\frac{1}{4}} = 1$$

We remove area 1

- Register for turnitin.com by WEDNESDAY, 8/13
  - > Class ID: 8299378
  - > Enrollment Password: vismath
- Start doing research for one-page paper on Fractals - due FRIDAY, 8/15
  - > watch NOVA documentary
  - > look through articles in Brewer's Google Drive folder
  - > browse articles in the searchable Bridges archive
  - > look at books in library & S201
    - – In the ASMS Library:
      - « Benoit Mandelbrot's *Fractal Geometry of Nature*
      - « Mandelbrot's *The (mis)behavior of markets: a fractal view of risk, ruin, and reward*
      - « *Chaos and fractals: new frontiers of science*
    - – In S201:
      - « *Chaos and Fractals: The Mathematics Behind the Computer Graphics*
      - « *Fractals, Chaos, Power Laws*
      - « *Chaos, Fractals, and Dynamics: Computer Experiments in Mathematics*
      - « *Viewpoints: Mathematical Perspective and Fractal Geometry in Art*
  - > search databases in the Alabama Virtual Library
    - – EBSCOhost
    - – InfoTrac
    - – Gale Virtual Reference Library
- Work on Fractal written problems - due MONDAY, 8/18