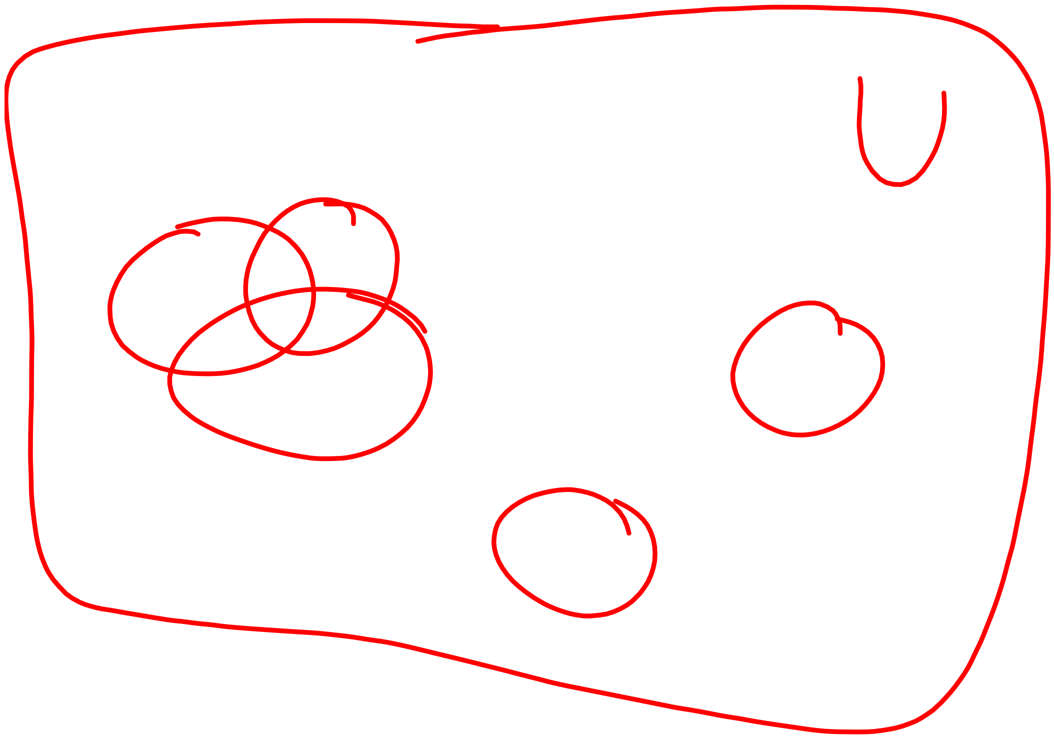
The "universal set" U is the set of all sets of the type of set you are dealing with.



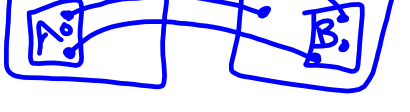
$f: A \to B$; $A_0 \subseteq A$; $B_0 \subseteq B$

The image of A_0 under f

$$f(A_0) = \{ b \in B \mid b = f(a) \text{ for some } a \in A_0 \}$$

The preimage of B_0 under f

$$f^{-1}(B_0) = \{ a \in A \mid f(a) \in B_0 \}$$



$f(f^{-1}(B_0)) \subseteq B_0$ (& $\supseteq$ if f is onto)

Proof: Let $y \in f(f^{-1}(B_0))$

$f(f^{-1}(B_0))$ is the image of $f^{-1}(B_0)$ under f.

$y \in f(f^{-1}(B_0)) \Rightarrow \exists a \in f^{-1}(B_0)$ s.t. $f(a) = y$

$f^{-1}(B_0)$ is the preimage of B_0 under f.

$a \in f^{-1}(B_0) \Rightarrow f(a) \in B_0$.

$f(a) = y \Rightarrow y \in B_0$. $\square$

$f(f^{-1}(B_0)) \subseteq B_0$ (& $\supseteq$ if f is onto)

$\supseteq$ Let $x \in B_0$

Since $f: A \to B$ is onto, and $B_0 \subset B$

$\exists a \in A$ s.t. $f(a) = x$.

Since $f(a) = x$ and $x \in B_0$,
we have $f(a) \in B_0$.

$f(a) \in B_0 \Rightarrow a \in f^{-1}(B_0)$

$\langle a \in A \Rightarrow f(a) \in f(A) \rangle$

$a \in f^{-1}(B_0) \Rightarrow f(a) \in f(f^{-1}(B_0))$

Since $x = f(a)$, we have $x \in f(f^{-1}(B_0))$.

2. $f: A \to B$; $A_i \subseteq A$; $B_i \subseteq B$

e. $A_0 \subseteq A_1 \implies f(A_0) \subset f(A_1)$

Proof: Given: $A_0 \subseteq A_1$

To show: $f(A_0) \subseteq f(A_1)$.

Let $x \in f(A_0)$. Since x is in the image of A_0 under f, we know that there exists some $a \in A_0$ such that $f(a) = x$.
Since $A_0 \subseteq A_1$, we have $a \in A_1$.
Since $a \in A_1$, we have $f(a) \in f(A_1)$.
Since $x = f(a)$, we have $x \in f(A_1)$. $\square$

$f: A \to B$, $A_0, A_1 \subseteq A$

2f. $f(A_0 \cup A_1) = f(A_0) \cup f(A_1)$

[To prove $=$, we need to prove $\subseteq$ & $\supseteq$]

Proof: $\subseteq$: Let $x \in f(A_0 \cup A_1)$.

$\Rightarrow \exists\, a \in A_0 \cup A_1$ such that $f(a) = x$.

$a \in A_0 \cup A_1 \Rightarrow a \in A_0$ or $a \in A_1$

Case 1: $a \in A_0 \Rightarrow f(a) \in f(A_0)$

$f(A_0) \subseteq f(A_0) \cup f(A_1)$.

$\Rightarrow f(a) \in f(A_0) \cup f(A_1)$.

$x = f(a) \Rightarrow x \in f(A_0) \cup f(A_1)$.

Case 2: $a \in A_1 \Rightarrow f(a) \in f(A_1)$

$f(A_1) \subseteq f(A_0) \cup f(A_1)$

$\Rightarrow f(a) \in f(A_0) \cup f(A_1)$

$x = f(a) \Rightarrow x \in f(A_0) \cup f(A_1)$.