2f. $f(A_0 \cup A_1) = f(A_0) \cup f(A_1)$

$\subseteq$ (proved Tuesday)

$\supseteq$: Let $y \in f(A_0) \cup f(A_1)$.

$\Rightarrow$ y is in the image of A_0 or y is in the image of A_1.

Case 1: $y \in f(A_0) \Rightarrow \exists\, a \in A_0$ s.t. $y = f(a)$.
$A_0 \subseteq A_0 \cup A_1 \Rightarrow a \in A_0 \cup A_1$
$\Rightarrow f(a) \in f(A_0 \cup A_1) \Rightarrow y \in f(A_0 \cup A_1)$.

Case 2: $y \in f(A_1) \Rightarrow \exists\, b \in A_1$ s.t. $y = f(b)$.
$A_1 \subseteq A_0 \cup A_1 \Rightarrow b \in A_0 \cup A_1$
$\Rightarrow f(b) \in f(A_0 \cup A_1) \Rightarrow y \in f(A_0 \cup A_1)$.

Hence $f(A_0) \cup f(A_1) \subseteq f(A_0 \cup A_1)$. $\square$

1.2
29. $f(A_0 \cap A_1) \subset f(A_0) \cap f(A_1)$
(& equality holds if f is injective)

Proof:

$\subseteq$: Let $x \in f(A_0 \cap A_1)$.

Since x is in the image of $A_0 \cap A_1$, $\exists\, a \in A_0 \cap A_1$ such that $x = f(a)$.
$a \in A_0 \cap A_1 \Rightarrow a \in A_0$ and $a \in A_1$.
$a \in A_0 \Rightarrow f(a) \in f(A_0)$ and
$a \in A_1 \Rightarrow f(a) \in f(A_1)$.
$\Rightarrow f(a) \in f(A_0)$ and $f(a) \in f(A_1)$
$\Rightarrow f(a) \in f(A_0) \cap f(A_1)$
$x = f(a) \Rightarrow x \in f(A_0) \cap f(A_1)$.
Hence $f(A_0 \cap A_1) \subseteq f(A_0) \cap f(A_1)$.

1.2 29. $f(A_0 \cap A_1) \subset f(A_0) \cap f(A_1)$
(& equality holds if f is injective)

to get equality all we need now is . . .

$\supseteq$: Suppose, in addition, f is injective.
Let $y \in f(A_0) \cap f(A_1)$.
$\Rightarrow y \in f(A_0)$ and $y \in f(A_1)$.
$y \in f(A_0) \Rightarrow \exists\, a \in A_0$ s.t. $y = f(a)$.
$y \in f(A_1) \Rightarrow \exists\, b \in A_1$ s.t. $y = f(b)$.
We have $y = f(a)$ and $y = f(b)$, so $f(a) = f(b)$. Since f is injective (one-to-one), this implies that $a = b$.
$a = b$ and $b \in A_1 \Rightarrow a \in A_1$.
$a \in A_0$ and $a \in A_1 \Rightarrow a \in A_0 \cap A_1$
$\Rightarrow f(a) \in f(A_0 \cap A_1)$
Since $y = f(a)$, we have $y \in f(A_0 \cap A_1)$.
Hence, $f(A_0) \cap f(A_1) \subseteq f(A_0 \cap A_1)$.
$\supseteq$ & $\subseteq$ $\Rightarrow$ the two sets are equal if f is injective! □
(images preserve intersections)

1.2 $f: A \to B$

3b. $f^{-1}\left(\bigcup_{B \in \mathcal{B}} B\right) = \bigcup_{B \in \mathcal{B}} f^{-1}(B)$

Proof:

$\subseteq$: Let $x \in f^{-1}\left(\bigcup_{B \in \mathcal{B}} B\right)$.

x in the preimage of $\bigcup_{B \in \mathcal{B}} B$ guarantees the existence of some/at least one element $b \in \bigcup_{B \in \mathcal{B}} B$ such that $f(x) = b$.

[also could be stated $f(x) \in \bigcup_{B \in \mathcal{B}} B$]

$b \in \bigcup_{B \in \mathcal{B}} B \Rightarrow b \in B$ for at least one $B \in \mathcal{B}$.

$\Rightarrow f(x) \in B \Rightarrow x \in f^{-1}(B)$.

$f^{-1}(B) \subseteq \bigcup_{B \in \mathcal{B}} f^{-1}(B) \Rightarrow x \in \bigcup_{B \in \mathcal{B}} f^{-1}(B)$.

Hence $f^{-1}\left(\bigcup_{B \in \mathcal{B}} B\right) \subseteq \bigcup_{B \in \mathcal{B}} f^{-1}(B)$.