1.2 - Functions

2b. $f^{-1}(B_0 \cup B_1) = f^{-1}(B_0) \cup f^{-1}(B_1)$

3b. $f^{-1}\left(\bigcup_{B \in \mathcal{B}} B\right) = \bigcup_{B \in \mathcal{B}} f^{-1}(B)$

Proof:

$\subseteq$ Let $x \in f^{-1}\left(\bigcup_{B \in \mathcal{B}} B\right)$.

x in the preimage of $\bigcup_{B \in \mathcal{B}} B$ implies that $f(x) \in \bigcup_{B \in \mathcal{B}} B$ (by definition of preimage).

$f(x) \in \bigcup_{B \in \mathcal{B}} B \Rightarrow f(x) \in B$ for at least one $B \in \mathcal{B}$.

$f(x) \in B \Rightarrow x \in f^{-1}(B)$ for at least one $B \in \mathcal{B}$.

$f^{-1}(B) \subseteq \bigcup_{B \in \mathcal{B}} f^{-1}(B) \Rightarrow x \in \bigcup_{B \in \mathcal{B}} f^{-1}(B)$.

Hence $f^{-1}\left(\bigcup_{B \in \mathcal{B}} B\right) \subseteq \bigcup_{B \in \mathcal{B}} f^{-1}(B)$.

$$f^{-1}\left(\bigcup_{B \in \mathcal{B}} B\right) = \bigcup_{B \in \mathcal{B}} f^{-1}(B)$$

Proof:

$\supseteq$: Let $y \in \bigcup_{B \in \mathcal{B}} f^{-1}(B)$. $\Rightarrow y \in f^{-1}(B)$ for at least one $B \in \mathcal{B}$.

$y \in f^{-1}(B) \Rightarrow f(y) \in B$ for at least one $B \in \mathcal{B}$.

$B \subseteq \bigcup_{B \in \mathcal{B}} B \Rightarrow f(y) \in \bigcup_{B \in \mathcal{B}} B$

$f(y) \in \bigcup_{B \in \mathcal{B}} B \Rightarrow y \in f^{-1}\left(\bigcup_{B \in \mathcal{B}} B\right)$.

Hence $\bigcup_{B \in \mathcal{B}} f^{-1}(B) \subseteq f^{-1}\left(\bigcup_{B \in \mathcal{B}} B\right)$. $\square$

2g. $f(A_0 \cap A_1) \subset f(A_0) \cap f(A_1)$
(& equality holds if f is injective)

3g. $f\left(\bigcap_{A \in \mathcal{A}} A\right) \subseteq \bigcap_{A \in \mathcal{A}} f(A)$

Proof: Let $x \in f\left(\bigcap_{A \in \mathcal{A}} A\right)$.

x in the image of $\bigcap_{A \in \mathcal{A}} A$ implies that

$\exists\, a \in \bigcap_{A \in \mathcal{A}} A$ s.t. $x = f(a)$.

$a \in \bigcap_{A \in \mathcal{A}} A \Rightarrow a \in A$ for all $A \in \mathcal{A}$.

$a \in A\ \forall A \in \mathcal{A} \Rightarrow f(a) \in f(A)$ for all $A \in \mathcal{A}$.

$f(a) = x \Rightarrow x \in f(A)$ for all $A \in \mathcal{A}$.

$\Rightarrow x \in \bigcap_{A \in \mathcal{A}} f(A)$. Hence $f\left(\bigcap_{A \in \mathcal{A}} A\right) \subseteq \bigcap_{A \in \mathcal{A}} f(A)$.

$f\left(\bigcap_{A \in \mathcal{A}} A\right) \subseteq \bigcap_{A \in \mathcal{A}} f(A)$

Suppose, in addition, that f is injective (one-to-one)

$\supseteq$: Let $y \in \bigcap_{A \in \mathcal{A}} f(A) \Rightarrow y \in f(A)$ for all $A \in \mathcal{A}$

$y \in f(A) \Rightarrow$ For each $A \in \mathcal{A}$, there exists $a \in A$ s.t. $y = f(a)$.

$$\boxed{\square \in \bigcap_{A \in \mathcal{A}} A \Rightarrow f(\square) \in f\left(\bigcap_{A \in \mathcal{A}} A\right)}$$

what we need to work to

$a \in A \Rightarrow f(a) \in f(A)$ (since $y = f(a)$, $f(a) \in f(A)\ \forall A \in \mathcal{A}$)

f is injective if $f(a) = f(b)$ implies that $a = b$.

$x \in A \Rightarrow f(x) \in f(A)$

$y \in B \Rightarrow f^{-1}(y) \in f^{-1}(B)$

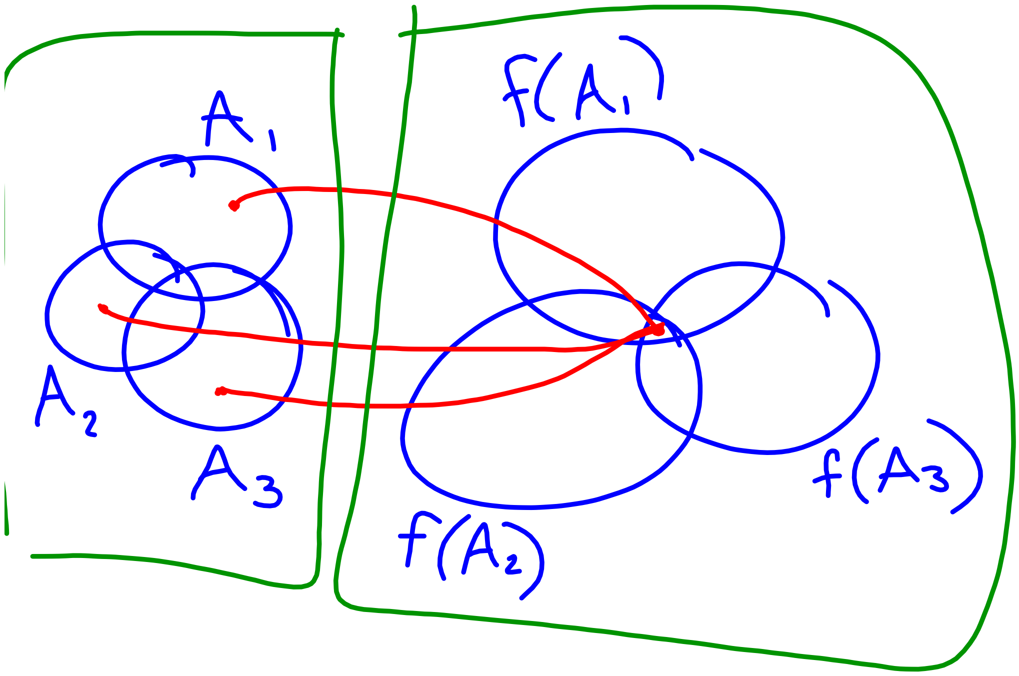
$y = f(a)$ $\quad a \in A$

$y \in f(A)$

$f^{-1}(y) \in f^{-1}(f(A)) = A$

We need to use injectivity to show that $f^{-1}(y) = a$

$\exists\, b \in A$ s.t. $f^{-1}(y) = b$ $\quad$ ... TBC Tues.


A_1
A_2
A_3
$f(A_1)$
$f(A_2)$
$f(A_3)$