$$f\left(\bigcap_{A\in\mathcal{A}} A\right) \subseteq \bigcap_{A\in\mathcal{A}} f(A)$$

Suppose, in addition, that f is injective (one-to-one).

$\supseteq$: Let $y \in \bigcap_{A_i\in\mathcal{A}} f(A_i) \Rightarrow y \in f(A_i)$ for all $A_i \in \mathcal{A}$

$\Rightarrow$ For each $A_i \in \mathcal{A}$, $\exists\ a_i \in A_i$ such that $y = f(a_i)$, i.e. $y = f(a_1)$ for $a_1 \in A_1$, $y = f(a_2)$ for $a_2 \in A_2$, etc.

That is, $f(a_i) = f(a_j)$ for all i, j.

Since f is injective,

$$f(a_i) = f(a_j) \Rightarrow a_i = a_j \text{ for all } i, j.$$

That is, $y = f(a)$ for $a \in \bigcap_{A_i\in\mathcal{A}} A_i$.

$a \in \bigcap_{A\in\mathcal{A}} A \Rightarrow f(a) \in f\left(\bigcap_{A\in\mathcal{A}} A\right)$

$y = f(a) \Rightarrow y \in f\left(\bigcap_{A\in\mathcal{A}} A\right)$

Hence, $\bigcap_{A\in\mathcal{A}} f(A) = f\left(\bigcap_{A\in\mathcal{A}} A\right)$.

(If f is injective). $\square$

1.2

Given $f: A \to B$ and $g: B \to C$,

$g \circ f: A \to C$ has the rule

$\star$ $\{(a,c) \mid$ For some $b \in B$, $f(a) = b$ and $g(b) = c\}$

Lemma: If a function has an inverse, then it is bijective.

1.2 ~~4~~. $f: A \to B$, $g: B \to C$

(a) If $C_0 \subseteq C$, show that
$$(g \circ f)^{-1}(C_0) = f^{-1}(g^{-1}(C_0)).$$

Proof

$\subseteq$: Let $x \in (g \circ f)^{-1}(C_0)$.

x in preimage of C_0 under $g \circ f$.

$\Rightarrow \exists\, c \in C_0$ s.t. $(g \circ f)(x) = c$.

$(g \circ f)(x) = c \Rightarrow \exists\, b \in B$ s.t. $f(x) = b$ and $g(b) = c$.

$g(b) = c$ & $c \in C_0 \Rightarrow b \in g^{-1}(C_0)$

$f(x) = b$ & $b \in g^{-1}(C_0)$

$\Rightarrow x \in f^{-1}(g^{-1}(C_0))$. $\square$

4. $f: A \to B$, $g: B - C$

(b) If f & g are injective, show that $g \circ f$ is injective.

Given: $f(a_1) = f(a_2) \Rightarrow a_1 = a_2 \quad \forall\, a_1, a_2 \in A$

$g(b_1) = g(b_2) \Rightarrow b_1 = b_2 \quad \forall\, b_1 \,\&\, b_2 \in B$

To show that $g \circ f$ is injective, i.e.

if $(g \circ f)(x_1) = (g \circ f)(x_2) \Rightarrow x_1 = x_2$.

Proof: Let $(g \circ f)(x_1) = (g \circ f)(x_2)$.

i.e. $g(f(x_1)) = g(f(x_2))$.

Since g is injective, we have $f(x_1) = f(x_2)$.

Since f is injective, we have $x_1 = x$. $\square$

2e. $A_0 \subset A_1 \Rightarrow f(A_0) \subset f(A_1)$

2a. $B_0 \subset B_1 \Rightarrow f^{-1}(B_0) \subset f^{-1}(B_1)$