2.12 Topological Spaces

Def: A topology on a set X is a collection $\mathcal{T}$ of subsets of X having the following properties:

(1) $\emptyset$ and X are in $\mathcal{T}$

(2) The union of the elements of any (arbitrary) subcollection of $\mathcal{T}$ is in $\mathcal{T}$

(3) The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$

A set X for which a topology $\mathcal{T}$ has been specified is called a topological space.

If X is a topological space with topology $\mathcal{T}$, we say that a subset U of X is open in X (or is an open set of X) if U belongs to the collection $\mathcal{T}$.

<u>finite complement topology</u>

Let X be a set; let τ_f be the collection of all subsets U of X such that $X-U$ is either finite or all of X.

$\emptyset$: $X - \emptyset = X$ (all of X ✓)

X: $X - X = \emptyset$ (finite ✓)

arbitrary unions:

$X - \bigcup U_\alpha =$

$U_\alpha \in \tau_f$ for all α

- If $U_\alpha = \emptyset$ $\forall \alpha$, $\bigcup U_\alpha = \emptyset \Rightarrow X - \emptyset = X$ ✓
- If $U_\alpha = X$ for any α, $\bigcup U_\alpha = X \Rightarrow X - X = \emptyset$ ✓
- $U_\alpha \neq \emptyset$ for at least one α & $U_\alpha \neq X$ $\forall \alpha$

$X - U_\alpha$ is finite for all α

$X - \bigcup U_\alpha \underset{\text{De Morgan's Law}}{=} \bigcap (X - U_\alpha)$

$\bigcap (X - U_\alpha) \subseteq (X - U_\alpha)$ for all α

Since $X - U_\alpha$ is finite, any subset of that is also finite. Hence, $\bigcup U_\alpha \in \tau_f$.

finite intersections:

$X - \bigcap_{i=1}^{n} U_i = \bigcup_{i=1}^{n} (X - U_i)$

- If $U_i = \emptyset$ for any i, $\bigcap_{i=1}^{n} U_i = \emptyset$; $X - \emptyset = X$
- If $U_i = X$ for all i, $\bigcap_{i=1}^{n} U_i = X$; $X - X = \emptyset$
- Suppose $U_i \neq \emptyset$ for all i and $U_i \neq X$ for at least one i.

Since $U_i \in \tau_f$, $X - U_i$ is finite.

$\bigcup_{i=1}^{n} (X - U_i)$ is a finite union of finite sets and Hence is finite. Thus $\bigcap U_i \in \tau$.

A set A is <u>countable</u> if there is an injective function $f: A \to \mathbb{N}$

$\mathbb{N} = \{1, 2, 3, \ldots\}$

All finite sets are cantable ★.

countably infinite v. uncountably infinite

$\mathbb{N}$ itself is infinite & countable

$\mathbb{Z}$ are countable

map negative's to even positive's

& positives to odd positives

$\mathbb{R}$ reals are uncountable

$\mathbb{Q}$ rationals are uncountable

irrationals $\mathbb{R}-\mathbb{Q}$ are uncountable

- union & intersection of finite sets is finite
- union of uncountable sets is uncountable
- union of infinite sets (countable or uncountable) is infinite

Standard topology on the real number line

$\mathcal{B}$ is the set of all open intervals

$$(a,b) = \{x \mid a < x < b\}$$

on the real line $\mathbb{R}$.

To show that $\mathcal{B}$ is a topology on $\mathbb{R}$.

$\emptyset = (a,a)$ for any $a \in \mathbb{R}$, so $\emptyset \in \mathcal{B}$

$\mathbb{R} = (-\infty, \infty)$ is open, so $\mathbb{R}$, so $\mathbb{R} \in \mathcal{B}$.

$\bigcup_\alpha (a_\alpha, b_\alpha)$ arbitrary union of open sets in open

$\bigcap_{i=1}^{n} (a_i, b_i)$ finite intersection of open sets is open

To be proved ...