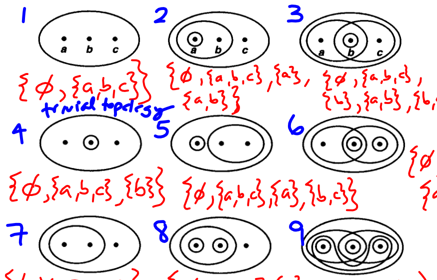
2. Consider the nine topologies on the set $X = \{a, b, c\}$ indicated in Example 1 of §12. Compare them; that is, for each pair of topologies, determine whether they are comparable, and if so, which is the finer.

$\mathcal{T}$ is comparable with $\mathcal{T}'$ if $\mathcal{T}' \supseteq \mathcal{T}$ or $\mathcal{T} \supseteq \mathcal{T}'$.

If $\mathcal{T}' \supseteq \mathcal{T}$, we say that $\mathcal{T}'$ is finer than $\mathcal{T}$, and that $\mathcal{T}$ is coarser than $\mathcal{T}'$.



1: $\{\phi, \{a,b,c\}\}$ trivial topology

2: $\{\phi, \{a,b,c\}, \{a\}, \{a,b\}\}$

3: $\{\phi, \{a,b,c\}, \{b\}, \{a,b\}, \{b,c\}\}$

Figure 12.1, page 76 topologies for the set {a,b,c}

$A \subseteq B$ A is a subset of B

$B \supseteq A$ B contains A

4: $\{\phi, \{a,b,c\}, \{b\}\}$

5: $\{\phi, \{a,b,c\}, \{a\}, \{b,c\}\}$

6: $\{\phi, X, \{b\}, \{c\}, \{a,b\}, \{b,c\}\}$

7: $\{\phi, X, \{a,b\}\}$

8: $\{\phi, X, \{a\}, \{b\}, \{a,b\}\}$

9: $\mathcal{P}(X)$ discrete topology

$1 \leq$ all $\Rightarrow$ 1 is the coarsest

$9 \geq$ all $\Rightarrow$ 9 is finest

$4 \leq 6$; $4 \leq 3$; $4 \leq 8$

$2 \leq 8$; $3 \leq 6$; $7 \leq 3$; $7 \leq 2$; $7 \leq 6$; $7 \leq 8$

$1 \leq 4 \leq 3 \leq 6 \leq 9$

$1 \leq 4 \leq 6 \leq 9$

$1 \leq 4 \leq 8 \leq 9$

$1 \leq 2 \leq 8 \leq 9$

$1 \leq 7 \leq 3 \leq 6 \leq 9$

$1 \leq 5 \leq 9$

$1 \leq 7 \leq 2 \leq 8 \leq 9$

Figure 12.2, page 77 NOT topologies for {a,b,c}

4. (a) If $\{\mathcal{T}_\alpha\}$ is a family of topologies on X, show that $\bigcap \mathcal{T}_\alpha$ is a topology on X. Is $\bigcup \mathcal{T}_\alpha$ a topology on X?

$\phi \in \bigcap \mathcal{T}_\alpha$? Yes

$\bigcap \mathcal{T}_\alpha \in \bigcap \mathcal{T}_\alpha$? Yes

$\bigcup U_\alpha \in \bigcap \mathcal{T}_\alpha$?, where $U_\alpha \subseteq \bigcap \mathcal{T}_\alpha$:

$U_\alpha \in \mathcal{T}_\alpha$ for all α.

$\mathcal{T}_\alpha$ is a topology on X for all α

$\Rightarrow \bigcup U_\alpha \in \mathcal{T}_\alpha$ for all α

$\Rightarrow \bigcup U_\alpha \in \bigcap \mathcal{T}_\alpha$ ✓

$\bigcap_{i=1}^{n} U_i \in \bigcap \mathcal{T}_\alpha$? , where $U_i \in \bigcap \mathcal{T}_\alpha$

$U_i \in \mathcal{T}_\alpha$ for all α

$\mathcal{T}_\alpha$ is a topology on X for all α

$\Rightarrow \bigcap_{i=1}^{n} U_i \in \mathcal{T}_\alpha$ for all α

$\Rightarrow \bigcap_{i=1}^{n} U_i \in \bigcap_{i=1}^{n} \mathcal{T}_\alpha$ ✓ $\square$

$\bigcup \tau_\alpha$ topology ?

$\phi \in \bigcup \tau_\alpha$ ✓

$\bigcup \tau_\alpha \in \bigcup \tau_\alpha$ ✓

$\bigcap_{i=1}^{n} U_i \overset{?}{\in} \bigcup \tau_\alpha$ for $U_i \in \bigcup \tau_\alpha$

$U_i \in \bigcup \tau_\alpha \Rightarrow U_i \in \tau_\alpha$ for at least one α.

τ_α is a topology $\Rightarrow \bigcap_{i=1}^{n} U_i \in \tau_\alpha$ for at least one α.

$\Rightarrow \bigcap_{i=1}^{n} U_i \in \bigcup \tau_\alpha$. ✓

$\bigcup U_\beta \overset{?}{\in} \bigcup \tau_\alpha$ for $U_\beta \in \bigcup \tau_\alpha$

$U_\beta \in \bigcup \tau_\alpha \Rightarrow U_\beta \in \tau_\alpha$ for at least one α

τ_α is a topology on $X \Rightarrow \bigcup U_\beta \in \tau_\alpha$ for at least one α. $\Rightarrow \bigcup U_\beta \in \bigcup \tau_\alpha$ ✓ $\square$

Yes, $\bigcup \tau_\alpha$ is a topology.

HW #4 (due next Wed. 04/16)

2.13 (page 83) #1,3,7,8