2.13

4(a) Given a family of topolgies on X, $\{\tau_\alpha\}$

$\bigcap \tau_\alpha$ is a topology on X

→ $\bigcup \tau_\alpha$ is not a topology on X (see counter-example in part (c))

$\phi \in \bigcap \tau_\alpha$, $\phi \in \bigcup \tau_\alpha$

$X \in \bigcap \tau_\alpha$, since $X \in \tau_\alpha$ for all α

$X \in \bigcup \tau_\alpha$, since $X \in \tau_\alpha$ for all α

4(b) $\{\tau_\alpha\}$ - Family of topologies on X

Show that there exists a unique smallest topology on X containing all collections τ_α, and a unique largest topology contained in all τ_α

(1) The smallest topology containing all τ_α is $\cancel{\bigcup \tau_\alpha}$

(2) The largest topology contained in all τ_α is $\bigcap \tau_\alpha$.

~~(1) Let τ_0 be a topology containing all the τ_α's.~~
~~$\tau_\alpha \subseteq \tau_0 \ \forall \alpha \Rightarrow$~~
~~$\bigcup \tau_\alpha \subseteq \tau_0$, i.e. either $\bigcup \tau_\alpha = \tau_0$ or~~
~~$\bigcup \tau_\alpha \subsetneq \tau_0$ and hence smaller than any such τ_0.~~

(2) Let τ_1 be a topology contained in all τ_α's.
$\tau_1 \subseteq \tau_\alpha \ \forall \alpha \Rightarrow$
$\tau_1 \subseteq \bigcap \tau_\alpha \Rightarrow \tau_1 = \bigcap \tau_\alpha$ or
$\tau_1 \subsetneq \bigcap \tau_\alpha$ and hence $\bigcap \tau_\alpha$ is larger than any such τ_1.

Instead of $\bigcup \mathcal{T}_\alpha$

we need

$$\bigcup \mathcal{T}_\alpha \cup (U_a \cap U_b) \; \forall \; U_a, U_b \in \mathcal{T}_\alpha \text{ for any } \alpha$$

Why is this the smallest topology on X containing all $\mathcal{T}_\alpha$'s?

Let $\mathcal{T}_0$ be a topology on X containing $\bigcup \mathcal{T}_\alpha$.
Since $\mathcal{T}_0$ is a topology, it contains all such intersections $U_a \cap U_b$ for $U_a, U_b \in \mathcal{T}_\alpha$ for any α, we have $\bigcup \mathcal{T}_\alpha \cup (U_a \cap U_b) \subseteq \mathcal{T}_0$,
so either $<$ or $=$.

(c) $X = \{a, b, c\}$

$\mathcal{T}_1 = \{\emptyset, X, \{a\}, \{a,b\}\}$

$\mathcal{T}_2 = \{\emptyset, X, \{a\}, \{b,c\}\}$

Smallest topology containing $\mathcal{T}_1$ and $\mathcal{T}_2$:

$\mathcal{T}_1 \cup \mathcal{T}_2 = \{\emptyset, X, \{a\}, \{a,b\}, \{b,c\}\}$

(not a topology!)

$\{\emptyset, X, \{a\}, \{b\}, \{a,b\}, \{b,c\}\}$ ☆ uh oh

largest topology contained in $\mathcal{T}_1$ and $\mathcal{T}_2$:

$\{\emptyset, X, \{a\}\}$ ✓ Good

okay

we like it

1. statement on p78

A subset U of X is said to be open in X (that is, an element of τ) if for each $x \in U$, there is a basis element $B \in \mathcal{B}$ s.t. $x \in B$ and $B \subseteq U$.

~~3~~ 7.

$$\tau_a \subseteq \tau_b$$

if given $U_a \in \tau_\alpha$ and $x \in U$, we can find some $U_b \in \tau_b$ s.t. $x \in U_b$ and $U_b \subseteq U_a$.