Definition. A ***topology*** on a set X is a collection $\mathcal{T}$ of subsets of X having the following properties:

(1) $\varnothing$ and X are in $\mathcal{T}$.

(2) The union of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

(3) The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

A set X for which a topology $\mathcal{T}$ has been specified is called a ***topological space***.

If X is a topological space with topology $\mathcal{T}$, we say that a subset U of X is an ***open set*** of X if U belongs to the collection $\mathcal{T}$. Using this terminology, one can say that a topological space is a set X together with a collection of subsets of X, called *open sets*, such that $\varnothing$ and X are both open, and such that arbitrary unions and finite intersections of open sets are open.

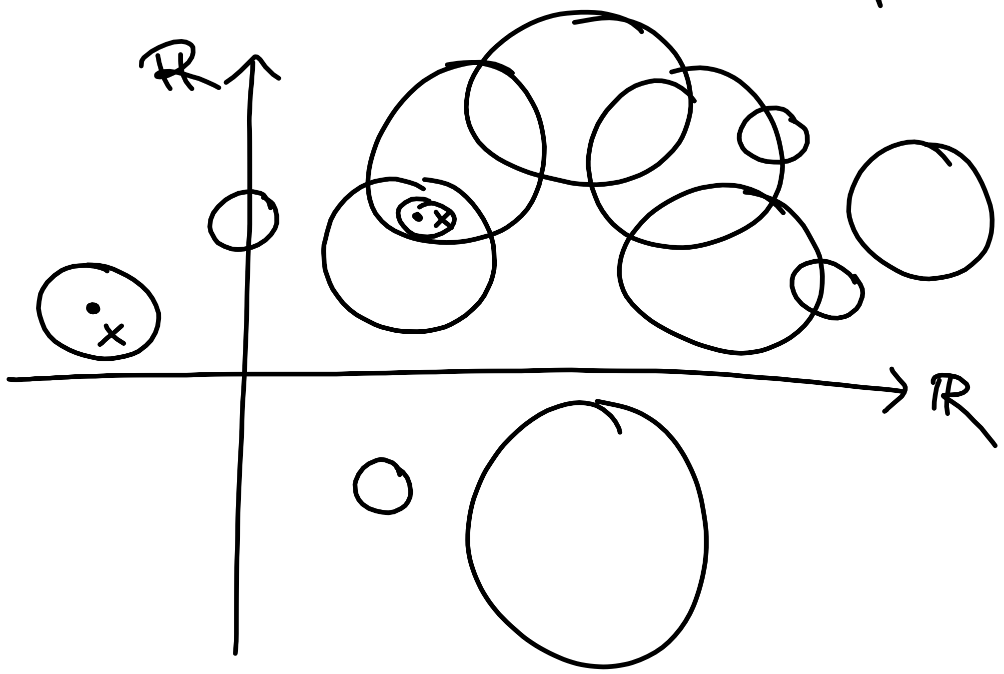
Definition. If X is a set, a ***basis*** for a topology on X is a collection $\mathcal{B}$ of subsets of X (called ***basis elements***) such that

(1) For each $x \in X$, there is at least one basis element B containing x.

(2) If x belongs to the intersection of two basis elements B_1 and B_2, then there is a basis element B_3 containing x such that $B_3 \subset B_1 \cap B_2$.

If $\mathcal{B}$ satisfies these two conditions, then we define the ***topology*** $\mathcal{T}$ ***generated by*** $\mathcal{B}$ as follows: A subset U of X is said to be open in X (that is, to be an element of $\mathcal{T}$) if for each $x \in U$, there is a basis element $B \in \mathcal{B}$ such that $x \in B$ and $B \subset U$. Note that each basis element is itself an element of $\mathcal{T}$.

$\mathbb{R} \times \mathbb{R}$ basis for $\mathcal{T}_{\mathbb{R}\times\mathbb{R}}$ = the set of all open circles



$$X = \{a, b, c\}$$

$$\mathcal{T} = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}\}$$

$$\cancel{B = \{\{a\}\}} \qquad \cancel{B = \{\{a\}, \{b\}\}}$$

$$b \in X$$

6. Show that the topologies of $\mathbb{R}_\ell$ and $\mathbb{R}_K$ are not comparable.

(show that neither is a subset of the other)

topology on $\mathbb{R}_\ell$ = "the lower limit topology"

$\mathcal{T}_\ell$ $\quad [a,b) = \{x \in \mathbb{R} \mid a \le x < b\}$

topology on $\mathbb{R}_K$ = "the K-Topology"

$\mathcal{T}_K$ $\quad (a,b)$ together with $(a,b) - K$,
where $K = \{1/n \mid n \in \mathbb{Z}_+\}$

$\mathcal{T}_1 \not\subseteq \mathcal{T}_2$: Take some $U_1 \in \mathcal{T}_1$ and $x \in U_1$

$\underline{\mathcal{T}_\ell \not\subseteq \mathcal{T}_K}$: $[0,2) \in \mathcal{T}_\ell$ and $0 \in [0,2)$,

for $\mathcal{T}_1 \subseteq \mathcal{T}_2$ we would need some $U_2 \in \mathcal{T}_2$ s.t.
$x \in U_2$ and $U_2 \subseteq U_1$.

but there is no such (a,b) or $(a,b)-K$ in $\mathcal{T}_K$
such that $0 \in (a,b)$ and $(a,b) \subseteq [0,2)$.

$\underline{\mathcal{T}_K \not\subseteq \mathcal{T}_\ell}$: $(-1,1)-K \in \mathcal{T}_K$ and $0 \in (-1,1)-K$

but there is no such $[a,b) \in \mathcal{T}_\ell$ such that
$0 \in [a,b)$ and $[a,b) \subseteq (-1,1)-K$.

7. Consider the following topologies on $\mathbb{R}$:

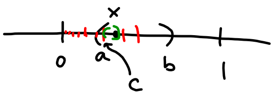
$\mathcal{T}_1$ = the standard topology, (a,b)
$\mathcal{T}_2$ = the topology of $\mathbb{R}_K$, (a,b) + $(a,b)-K$
$\mathcal{T}_3$ = the finite complement topology, $\{U \mid X-U \text{ is finite or all of } X\}$
$\mathcal{T}_4$ = the upper limit topology, having all sets $(a, b]$ as basis,
$\mathcal{T}_5$ = the topology having all sets $(-\infty, a) = \{x \mid x < a\}$ as basis.

Determine, for each of these topologies, which of the others it contains.

$\mathcal{T}_2 \subsetneq \mathcal{T}_4$: the topology on $\mathbb{R}_K$ is strictly coarser than the upper limit topology.

$\subseteq$: Case 1 : Let $(a,b) \in \mathcal{T}_2$ and $x \in (a,b)$.
Then $(a, \frac{b-x}{2}] \in \mathcal{T}_4$ and $x \in (a, \frac{b-x}{2}] \subseteq (a,b)$.

Case 2 : Let $(a,b)-K \in \mathcal{T}_2$ and $x \in (a,b)-K$.
If $a > 1$, $(a,b)-K = (a,b)$ (similar to case 1)
If $b < 0$, $(a,b)-K = (a,b)$ (similar to case 1)
If $(a,b) \subseteq (0,1)$: Take c = the largest number of the form $\frac{1}{n}$ such that $x > c$.



Then $(c, x] \in \mathcal{T}_4$
$x \in (c,x]$ and
$(c,x] \subseteq (a,b)-K$.

$\not\supseteq$: Take $(-1, 0] \in \mathcal{T}_4$ and $0 \in (-1,0]$
But there is no (a,b) or $(a,b)-K$ containing 0 that lies inside $(-1,0]$.

8. (a) Apply Lemma 13.2 to show that the countable collection

$$\mathcal{B} = \{(a, b) \mid a < b, a \text{ and } b \text{ rational}\}$$

is a basis that generates the standard topology on $\mathbb{R}$.

(b) Show that the collection

$$\mathcal{C} = \{[a, b) \mid a < b, a \text{ and } b \text{ rational}\}$$

is a basis that generates a topology different from the lower limit topology on $\mathbb{R}$.

Lemma 13.2. *Let X be a topological space. Suppose that $\mathcal{C}$ is a collection of open sets of X such that for each open set U of X and each x in U, there is an element C of $\mathcal{C}$ such that $x \in C \subset U$. Then $\mathcal{C}$ is a basis for the topology of X.*