1. Let X be a topological space; let A be a subset of X. Suppose that for each $x \in A$ there is an open set U containing x such that $U \subset A$. Show that A is open in X.

Definition. If X is a set, a ***basis*** for a topology on X is a collection $\mathcal{B}$ of subsets of X (called ***basis elements***) such that

(1) For each $x \in X$, there is at least one basis element B containing x.

(2) If x belongs to the intersection of two basis elements B_1 and B_2, then there is a basis element B_3 containing x such that $B_3 \subset B_1 \cap B_2$.

If $\mathcal{B}$ satisfies these two conditions, then we define the ***topology*** $\mathcal{T}$ ***generated by*** $\mathcal{B}$ as follows: A subset U of X is said to be open in X (that is, to be an element of $\mathcal{T}$) if for each $x \in U$, there is a basis element $B \in \mathcal{B}$ such that $x \in B$ and $B \subset U$. Note that each basis element is itself an element of $\mathcal{T}$.

3. Show that the collection $\mathcal{T}_c$ given in Example 4 of §12 is a topology on the set X. Is the collection

$$\mathcal{T}_\infty = \{U \mid X - U \text{ is infinite or empty or all of } X\}$$

a topology on X? No. Find a counter-example where one of the criteria fails.

$\mathcal{T}_c$ = cocountable topology

$= \{U \mid X - U \text{ is countable or all of } X\}$

Need to show

$\emptyset \in \mathcal{T}_c$

$X \in \mathcal{T}_c$

$\bigcup U_\alpha \in \mathcal{T}_c$ for $U_\alpha \in \mathcal{T}_c$

$\bigcap_{i=1}^{n} U_i \in \mathcal{T}_c$ for $U_i \in \mathcal{T}_c$

Use de Morgan's Laws

$X - \bigcup U_\alpha = \bigcap (X - U_\alpha)$

7. Consider the following topologies on $\mathbb{R}$:

$\mathcal{T}_1$ = the standard topology, (a,b)
$\mathcal{T}_2$ = the topology of $\mathbb{R}_K$, $(a,b) + (a,b)-K$
$\mathcal{T}_3$ = the finite complement topology, $\{U \mid X-U \text{ is finite or all of } X\}$
$\mathcal{T}_4$ = the upper limit topology, having all sets $(a,b]$ as basis,
$\mathcal{T}_5$ = the topology having all sets $(-\infty, a) = \{x \mid x < a\}$ as basis.

Determine, for each of these topologies, which of the others it contains.

$\mathcal{T}_2 \subsetneq \mathcal{T}_4$

to show that $\mathcal{T}_a \subseteq \mathcal{T}_b$,
let $U \in \mathcal{T}_a$ and $x \in U$ and find $V \in \mathcal{T}_b$ s.t. $x \in V \subseteq U$.

Let $U \in \mathcal{T}_3$, $U \neq X$. Then $X-U$ is finite.
That is $X-U = \{a_1, \dots, a_n\}$ for $a_1, \dots a_n \in \mathbb{R}$.
$\Rightarrow U = (-\infty, a_1) \cup (a_1, a_2) \cup \dots \cup (a_n, \infty)$

8. (a) Apply Lemma 13.2 to show that the countable collection

$$\mathcal{B} = \{(a,b) \mid a < b,\ a \text{ and } b \text{ rational}\}$$

is a basis that generates the standard topology on $\mathbb{R}$.

(b) Show that the collection

$$\mathcal{C} = \{[a,b) \mid a < b,\ a \text{ and } b \text{ rational}\}$$

is a basis that generates a topology different from the lower limit topology on $\mathbb{R}$.

Lemma 13.2. *Let X be a topological space. Suppose that $\mathcal{C}$ is a collection of open sets of X such that for each open set U of X and each x in U, there is an element C of $\mathcal{C}$ such that $x \in C \subset U$. Then $\mathcal{C}$ is a basis for the topology of X.*

(a) Given (a,b) (an open set in the standard topology on $\mathbb{R}$)
and $x \in (a,b)$
we need $B \in \mathcal{B}$ s.t. $x \in B \subseteq (a,b)$.

(b) 1. show that the collection is a basis for a topology on $\mathbb{R}$ (using definition)

2. to show it's different from lower limit topology, try to compare the 2

$\mathcal{T}_1 \subseteq \mathcal{T}_2$??
$\mathcal{T}_2 \subsetneq \mathcal{T}_1$??

maybe one is finer, or not comparable at all

Closed Sets, Limit Points, and Continuity

Recall: U is open in X if $U \in \mathcal{T}_X$.

Def Let X be a topological space. $A \subset X$ is a **closed set** if $X - A$ ia open.

Examples:

1. $\emptyset$ is closed since $X - \emptyset = X$ is open
   X is closed since $X - X = \emptyset$ is open
   X & $\emptyset$ are both open and closed

2. $[a,b]$ in $\mathbb{R}$ (w/ standard topology)
   $\mathbb{R} - [a,b] = (-\infty, a) \cup (b, \infty)$
   Since $(-\infty, a)$, (b, ∞) both open,
   $(-\infty, a) \cup (b, \infty)$ open
   $\Rightarrow [a,b]$ is closed

Why is (b, ∞) open?
$\bigcup_{n=1}^{\infty} (b, b+n)$ arbitrary union of open intervals

3. $(-\infty, a]$ or $[a, \infty)$?
   $\mathbb{R} - (-\infty, a] = (a, \infty)$ open
   $\Rightarrow (-\infty, a]$ is closed

4. co-finite topology on X
   $\{U \mid X - U$ is finite ~~or all of~~ $X\}$
   $A = \{a\} \subseteq X$
   $X - A = X - \{a\}$
   $X - (X - \{a\}) = \{a\}$ finite
   $\Rightarrow X - \{a\}$ is open
   $\Rightarrow \{a\}$ is closed

Thm 17.1 Let X be a topological space. The following hold:

1) $\emptyset$, X are closed
2) arbitrary intersections of closed sets are closed, i.e. if A_i are closed, $\bigcap_i A_i$ is closed.
3) finite unions of closed sets are closed

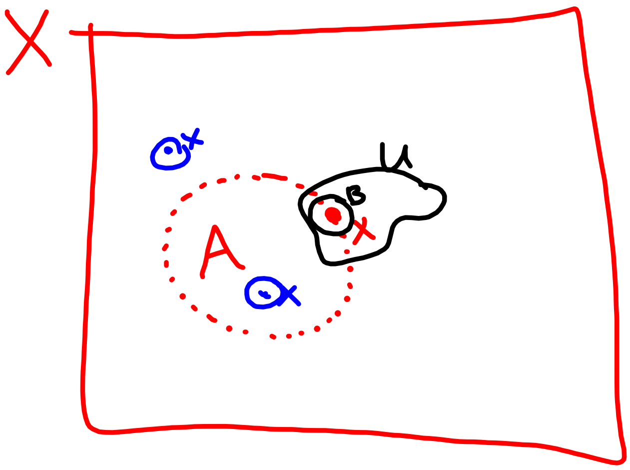
Proof: $\emptyset, X$ are closed as
$X - \emptyset = X$ and $X - X = \emptyset$ are open

2. $\bigcap A_i$ is closed if $X - \bigcap A_i$ is open
$X - \bigcap A_i = \bigcup (X - A_i)$
A_i closed $\forall i$, $X - A_i$ open
arbitrary union of open sets is open

3. $\bigcup_{i=1}^{n} A_i$ is closed if $X - \bigcup_{i=1}^{n} A_i$ is open
$X - \bigcup_{i=1}^{n} A_i = \bigcap_{i=1}^{n} X - A_i$.
A_i closed, $X - A_i$ open
finite intersection of open sets is open

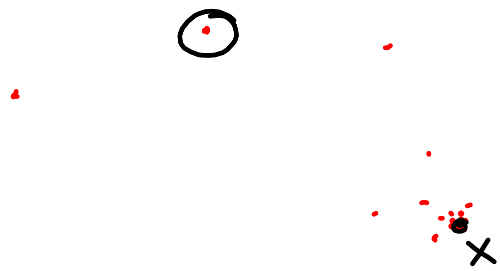
Def Let X be a topological space and let $A \subset X$. The **dosure** of A, denoted by $\bar{A}$, is the intersection of all closed sets containing A.

Thm 17.5 Let $A \subset X$ and let $\mathfrak{B}$ be a basis for X. Then $x \in \bar{A}$ if and only if every open set U containing x intersects A and $x \in \bar{A}$ if and only if every basis element B containing x intersects A



Def U is a **neighborhood** of x if U is open and $x \in U$

Def Let X be a topological space and let $A \subset X$. $x \in X$ is said to be a **limit point** of A if every neighborhood of x intersects A in a point other than x.



Thm 17.6 Let $A \subset X$ and let A' be the set of all limit points of A. Then $\bar{A} = A \cup A'$.

Cor 17.7 A is closed if and only if A contains all its limit points.

(a,b) limit points are a & b

$\overline{(a,b)} = [a,b]$

U = open disk $\overline{U}$ = closed disk