Definition. A ***topology*** on a set X is a collection $\mathcal{T}$ of subsets of X having the following properties:

(1) $\varnothing$ and X are in $\mathcal{T}$.

(2) The union of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

(3) The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

A set X for which a topology $\mathcal{T}$ has been specified is called a ***topological space***.

If X is a topological space with topology $\mathcal{T}$, we say that a subset U of X is an ***open set*** of X if U belongs to the collection $\mathcal{T}$. Using this terminology, one can say that a topological space is a set X together with a collection of subsets of X, called *open sets*, such that $\varnothing$ and X are both open, and such that arbitrary unions and finite intersections of open sets are open.

Definition. If X is a set, a ***basis*** for a topology on X is a collection $\mathcal{B}$ of subsets of X (called ***basis elements***) such that

(1) For each $x \in X$, there is at least one basis element B containing x.

(2) If x belongs to the intersection of two basis elements B_1 and B_2, then there is a basis element B_3 containing x such that $B_3 \subset B_1 \cap B_2$.

If $\mathcal{B}$ satisfies these two conditions, then we define the ***topology $\mathcal{T}$ generated by $\mathcal{B}$*** as follows: A subset U of X is said to be open in X (that is, to be an element of $\mathcal{T}$) if for each $x \in U$, there is a basis element $B \in \mathcal{B}$ such that $x \in B$ and $B \subset U$. Note that each basis element is itself an element of $\mathcal{T}$.

Lemma 13.2. *Let X be a topological space. Suppose that $\mathcal{C}$ is a collection of open sets of X such that for each open set U of X and each x in U, there is an element C of $\mathcal{C}$ such that $x \in C \subset U$. Then $\mathcal{C}$ is a basis for the topology of X.*

1. Let X be a topological space; let A be a subset of X. Suppose that for each $x \in A$ there is an open set U containing x such that $U \subset A$. Show that A is open in X.

3. Show that the collection $\mathcal{T}_c$ given in Example 4 of §12 is a topology on the set X. Is the collection

$$\mathcal{T}_\infty = \{U \mid X - U \text{ is infinite or empty or all of } X\}$$

a topology on X?

7. Consider the following topologies on $\mathbb{R}$:

$\mathcal{T}_1$ = the standard topology,
$\mathcal{T}_2$ = the topology of $\mathbb{R}_K$,
$\mathcal{T}_3$ = the finite complement topology,
$\mathcal{T}_4$ = the upper limit topology, having all sets $(a, b]$ as basis,
$\mathcal{T}_5$ = the topology having all sets $(-\infty, a) = \{x \mid x < a\}$ as basis.

Determine, for each of these topologies, which of the others it contains.

8. (a) Apply Lemma 13.2 to show that the countable collection

$$\mathcal{B} = \{(a, b) \mid a < b, a \text{ and } b \text{ rational}\}$$

is a basis that generates the standard topology on $\mathbb{R}$.

(b) Show that the collection

$$\mathcal{C} = \{[a, b) \mid a < b, a \text{ and } b \text{ rational}\}$$

is a basis that generates a topology different from the lower limit topology on $\mathbb{R}$.

1. Let X be a topological space; let A be a subset of X. Suppose that for each $x \in A$ there is an open set U containing x such that $U \subset A$. Show that A is open in X.

from def'n of a basis:

follows: A subset U of X is said to be open in X (that is, to be an element of $\mathcal{T}$) if for each $x \in U$, there is a basis element $B \in \mathcal{B}$ such that $x \in B$ and $B \subset U$. Note that

sketch of plan:

A is open in X if $A \in \mathcal{T}_X$

Every topology has a basis $\mathcal{B}$.

According to this,

A is open in X if for each $x \in A$, $\exists B \in \mathcal{B}$ s.t $x \in B$ and $B \subseteq A$.

Given: X, a topological space w/ topology $\mathcal{T}$

$A \subseteq X$

$\forall x \in A$, $\exists U \in \mathcal{T}$ s.t. $x \in U$ and $U \subseteq A$

Proof: Let $x \in A$. By assumption, $\exists U \in \mathcal{T}$ s.t. $x \in U$ and $U \subseteq A$.

Since U is open in X, $\exists B \in \mathcal{B}$ s.t ...

3. Show that the collection $\mathcal{T}_c$ given in Example 4 of §12 is a topology on the set X. Is the collection

$$\mathcal{T}_\infty = \{U \mid X - U \text{ is infinite or empty or all of } X\}$$

a topology on X?

$\emptyset : X - \emptyset = X \Rightarrow \emptyset \in \mathcal{T}_\infty$

$X : X - X = \emptyset \Rightarrow X \in \mathcal{T}_\infty$

$\bigcap_{i=1}^{n} U_i : X - \bigcap_{i=1}^{n} U_i = \bigcup_{i=1}^{n} (X - U_i)$

$U_i \in \mathcal{T}_\infty \ \forall i$,

1. If $U_i = \emptyset$ for any i, $X - U_i = X \subseteq \bigcup_{i=1}^{n}(X - U_i)$
2. If $U_i = X$ for all i, $X - U_i = \emptyset$
3. $U_i \neq \emptyset$ for all i and $U_i \neq X$ for at least one i

   $X - U_i$ is infinite for at least one i

   $X - U_i \subseteq \bigcup_{i=1}^{n} (X - U_i)$ is infinite

   $\Rightarrow \bigcap_{i=1}^{n} U_i \in \mathcal{T}_\infty$

$\bigcup_\alpha U_\alpha : X - \bigcup_\alpha U_\alpha = \bigcap_\alpha (X - U_\alpha)$

1. If $U_\alpha = X$ for at least one α, $X - U_\alpha = \emptyset$
2. If $U_\alpha = \emptyset$ for all α $X - U_\alpha = X$
3. $U_\alpha \neq X$ for any α and $U_\alpha \neq \emptyset$ for at least one α

   $\Rightarrow X - U_\alpha$ is infinite

   $\bigcap_\alpha (X - U_\alpha) \subseteq (X - U_\alpha)$

   inconclusive

$\cancel{\mathbb{R}}$

$\mathbb{Z} = X$

$\mathbb{Z}_+$

$\mathbb{Z}_-$

$\mathbb{Z}_+ \cup \{0\}$

$\mathbb{Z}_- \cup \{0\}$

Construct 2 subsets of $\mathbb{Z}$ U_1 & U_2 s.t. $X - U_1$ & $X - U_2$ are infinite (or empty or all of X) but $X - (U_1 \cup U_2)$ is finite.

$(X-U_1) \cap (X-U_2)$

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$\mathcal{T}_4$ = the upper limit topology, having all sets $(a, b]$ as basis,
$\mathcal{T}_5$ = the topology having all sets $(-\infty, a) = \{x \mid x < a\}$ as basis.

Determine, for each of these topologies, which of the others it contains.

$\tau_4 \not\subseteq \tau_5$?

Let $(a,b] \in \tau_4$ and $x \in (a,b]$.

We need some $(-\infty, c) \in \tau_5$ s.t. $x \in (-\infty, c)$ and $(-\infty, c) \subseteq (a,b]$.

Let $(0,1] \in \tau_4$ and $1 \in (0,1]$.

$\nexists\ (-\infty, a) \in \tau_5$ s.t. $1 \in (-\infty, a) \subseteq (0,1]$.

$\tau_5 \subseteq \tau_4$?

Let $(-\infty, a) \in \tau_5$ and $x \in (-\infty, a)$.

We need some $(b,c] \in \tau_4$ s.t. $x \in (b,c] \subseteq (-\infty, a)$.

$x \in (x-1, x] \subseteq (-\infty, a)$

$\tau_5 \subsetneq \tau_4$

7. Consider the following topologies on $\mathbb{R}$:

$\mathcal{T}_1$ = the standard topology,
$\mathcal{T}_2$ = the topology of $\mathbb{R}_K$, $(a,b) \cup (a,b)-\{\frac{1}{n}, n\in\mathbb{Z}_+\}$
$\mathcal{T}_3$ = the finite complement topology, $\{U \mid \mathbb{R}-U$ is finite or all of $\mathbb{R}\}$
$\mathcal{T}_4$ = the upper limit topology, having all sets $(a, b]$ as basis,
$\mathcal{T}_5$ = the topology having all sets $(-\infty, a) = \{x \mid x < a\}$ as basis.

Determine, for each of these topologies, which of the others it contains.

$U \in \mathcal{T}_3$ looks like:

Since $\mathbb{R}-U$ is finite, suppose
$\mathbb{R}-U = \{x_1, x_2, \ldots, x_n\}$, $x_1, \ldots, x_n \in \mathbb{R}$
$U = (-\infty, x_1) \cup (x_1, x_2) \cup \cdots \cup (x_n, \infty)$

$\mathcal{T}_2 \not\subseteq \mathcal{T}_3$?
Case1 : Let $(a,b) \in \mathcal{T}_2$ and $x \in (a,b)$
Case2 : Let $(a,b)-K \in \mathcal{T}_2$
Let $(0,1) \in \mathcal{T}_2$ and $\frac{1}{2} \in (0,1)$ $\nexists U \in \mathcal{T}_3$
s.t. $\frac{1}{2} \in U \subseteq (0,1)$

$\mathcal{T}_3 \subseteq \mathcal{T}_2$? Let $U \in \mathcal{T}_3$ and $x \in U$.
$x \in (-\infty, a)$ or $x \in (b,c)$ or $x \in (d, \infty)$
$x \in (x-1, \frac{a-x}{2})$ or $(\frac{x-b}{2}, \frac{c-x}{2})$ or $(\frac{x-d}{2}, x+1)$

$\Rightarrow \mathcal{T}_3 \subsetneq \mathcal{T}_2$

8. (a) Apply Lemma 13.2 to show that the countable collection

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is a basis that generates the standard topology on $\mathbb{R}$.

(b) Show that the collection

$$\mathcal{C} = \{[a, b) \mid a < b,\ a \text{ and } b \text{ rational}\}$$

is a basis that generates a topology different from the lower limit topology $\mathcal{T}_\ell$ on $\mathbb{R}$.

$\mathcal{C} \subseteq \mathcal{T}_\ell$?
$\mathcal{T}_\ell \subseteq \mathcal{C}$?

(a) Let $x \in \mathbb{R}$
need $B \in \mathcal{B}$ s.t. $x \in B$
Given B_1 & $B_2 \in \mathcal{B}$, $x \in B_1 \cap B_2$,
we need $B_3 \subseteq B_1 \cap B_2$ s.t. $x \in B_3$.