Closed Sets, Limit Points, and Continuity

Def Let X be a topological space. $\mathrm{A} \subset X$ is a **closed set** if $X - A$ ia open.

Thm 17.1 Let X be a topological space. The following hold:

1) $\emptyset$, X are closed
2) arbitrary intersections of closed sets are closed, i.e. if A_i are closed, $\bigcap_i A_i$ is closed.
3) finite unions of closed sets are closed

Def Let X be a topological space and let $\mathrm{A} \subset X$. The **dosure** of A, denoted by $\bar{A}$, is the intersection of all closed sets containing A.

Thm 17.5 Let $\mathrm{A} \subset X$and let $\mathfrak{B}$ be a basis for X. Then $x \in \bar{A}$ if and only if every open set U containing x intersects A and $x \in \bar{A}$ if and only if every basis element B containing x intersects A

Def U is a **neighborhood** of x if U is open and $x \in U$

Def Let X be a topological space and let $\mathrm{A} \subset X$. $x \in X$ is said to be a **limit point** of A if every neighborhood of x intersects A in a point other than x.

Thm 17.6 Let $\mathrm{A} \subset X$ and let A' be the set of all limit points of A. Then $\bar{A} = A \cup A'$.

Cor 17.7 A is closed if and only if A contains all its limit points.

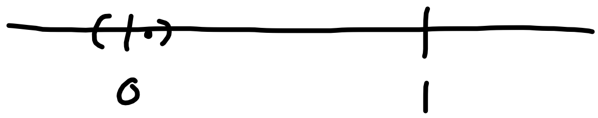
$\bar{A}$ is closed. intersection of closed sets.

$$\bar{A} = \bigcap_{A \subseteq C} C \quad , C \text{ closed}$$

$A \subseteq \bar{A}$

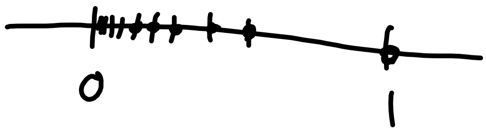
$\bar{A} = A$ iff A is closed

$A = (0, 1]$



$\overline{A} = [0, 1]$

$B = \{ \frac{1}{n} \mid n \in \mathbb{Z}_+ \}$

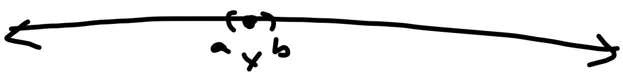


$\overline{B} = \cancel{[0, \infty)} = \cancel{[0, 1]}$

$= B \cup \{0\}$

$\mathbb{Q} = \{ p/q \mid p, q \in \mathbb{Z} \}$ = the set of rational #'s

$\overline{\mathbb{Q}} = \mathbb{R}$



every open interval of real #'s contains a rational #

$\mathbb{Z}$ = the set of integers

$\overline{\mathbb{Z}} = \mathbb{Z}$ $\quad$ $\mathbb{Z}$ has no limit points

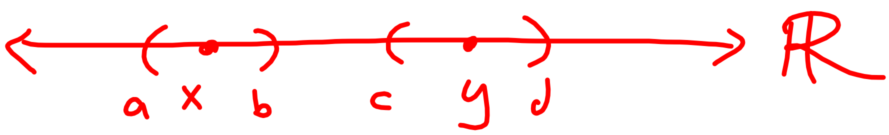
Thm 17.6 Let $A \subset X$ and let A' be the set of all limit points of A. Then $\bar{A} = A \cup A'$.

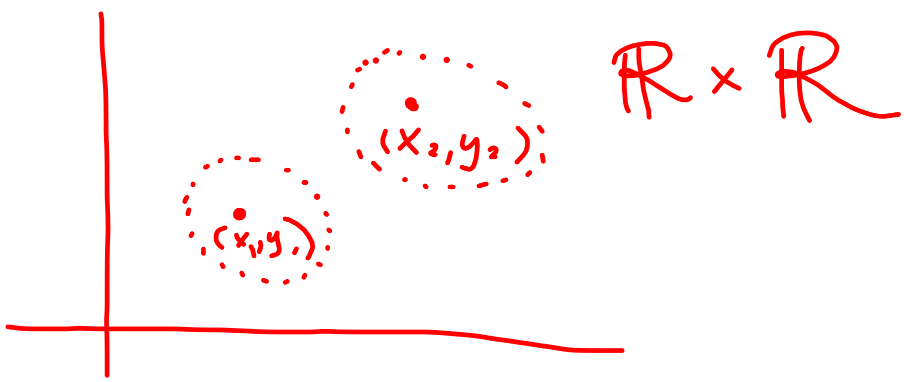
Cor 17.7 A is closed if and only if A contains all its limit points.

Proof: A is closed $\Rightarrow A = \bar{A} = A \cup A'$
$\Rightarrow A' \subseteq A \cup A' = A \Rightarrow A$ contains all its limit points

$\Leftarrow$ A contains all its limit points
$\Rightarrow A' \subseteq A \Rightarrow A = A \cup A' = \bar{A}$
$A = \bar{A} \Rightarrow A$ is closed!

Def A topological space is **Hausdorff** if for each pair of distinct points x and y, there exist disjoint neighborhoods of x and y.





X with the indiscrete topology (trivial)

$\tau = \{\emptyset, X\}$

clearly not Hausdorff

Cofinite topology on $\mathbb{Z}_+$

$\{U \mid \mathbb{Z}_+ - U \text{ is finite or all of } \mathbb{Z}_+\}$

$x = 1$, $y = 2$ $\in \mathbb{Z}_+$

not Hausdoff