Product Topology

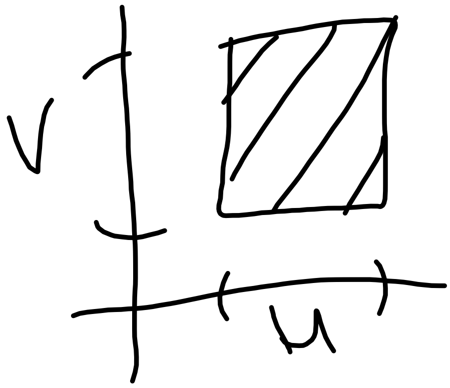
X & Y

$\mathcal{B}$ & $\mathcal{C}$

$X \times Y$

$\mathcal{B} \times \mathcal{C}$

$U \times V$



Basis

Theorem 15.1 $\mathcal{B}$ is basis of top. on X and $\mathcal{C}$ is basis of top. on Y, $\mathcal{B} \times \mathcal{C}$ is basis for product top. on $X \times Y$.

<u>Proof:</u> Given $X \times Y$ is a product top. $(x,y) \in X \times Y$. Show that $\exists\, B \times C \in \mathcal{B} \times \mathcal{C}$ st $(x,y) \in B \times C$.

$\Rightarrow x \in X$ and $y \in Y$.

$\Rightarrow \exists\, B \in \mathcal{B}$ st. $x \in B$.

Similarly, $\exists\, C \in \mathcal{C}$ st $y \in C$.

Hence, $(x,y) \in (B \times C)$.

Let $(x,y) \in (B_1 \times C_1) \cap (B_2 \times C_2)$

$\Rightarrow (x,y) \in B_1 \times C_1$ and $(x,y) \in B_2 \times C_2$.

$\Rightarrow x \in B_1$ and $x \in B_2$

$\Rightarrow x \in B_1 \cap B_2$

$\Rightarrow x \in B_3 \subset B_1 \cap B_2$

Similarly, $y \in C_3 \subset C_1 \cap C_2$.

$\Rightarrow (x,y) \in (B_3 \times C_3) \subset (B_1 \times C_1) \cap (B_2 \times C_2)$.

$\square$

$f: X \to Y$ is an open map if for every open set U in X, $f(U)$ is open in Y.

$\pi_1 : (X \times Y) \to X$ and
$\pi_2 (X \times Y) \to Y$ are open maps.

Proof π_1:

Let $U \times V$ be open in $X \times Y$.
$(x, y) \in U \times V$
$X \times Y$ has basis $\mathcal{B} \times \mathcal{C}$. st
$\exists\, B \times C \in \mathcal{B} \times \mathcal{C}$ st $(x, y) \in B \times C$.
$\pi_1(x, y) = x \in B \in \mathcal{B}$ st $x \in B \subset U$
Hence U is open.

Similarly, there exists a C in $\mathcal{C}'$ st $y \in C \subset V$, ie V is open.