Subspace Toplogy

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Definition:

Let X be a topological space with a topology T.
If Y is a subset of X, the collection

$$T_Y = \{ Y \cap U \mid U \in T \}$$

is a toplogy on Y, called the subspace topology.
With this topoogy Y is a subset of X, so its open sets consist of all intersections of open sets of X with Y.

$\emptyset : Y \cap \emptyset = \emptyset \quad \emptyset \in \mathcal{T}$

$Y : Y \cap X = Y \quad X \in \mathcal{T}$

$\bigcap_{i=1}^{n} (U_i \cap Y)$: $U_i = \emptyset$ foratleastonei

$\bigcap_{i=1}^{n} (U_i \cap Y) = \emptyset \quad \emptyset \in \mathcal{T}$

if $U_i = X \ \forall i$ $\bigcap_{i=1}^{n} (U_i \cap Y) = Y \quad U_i \in \mathcal{T}$

if $U_i \neq \emptyset \ \forall i$ and $U_i \neq X$ foratleastonei

Since $U_i \in \mathcal{T} \Rightarrow U_i$ open X

$\Rightarrow \bigcap_{i=1}^{n} U_i$ open X

$\bigcap_{i=1}^{n} (U_i \cap Y) = \left(\bigcap_{i=1}^{n} U_i\right) \cap Y \Rightarrow$ open

$\bigcup_{\alpha} (U_\alpha \cap Y)$: $U_\alpha = X$ foratleastonα

$\bigcup_{\alpha} (U_\alpha \cap Y) = Y \quad U_\alpha \in \mathcal{T}$

$U_\alpha = \emptyset \ \forall_\alpha \quad \bigcup_{\alpha} (U_\alpha \cap Y) = \emptyset \quad U_\alpha \in \mathcal{T}$

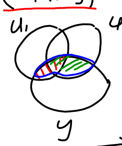
if $U_\alpha \neq \emptyset \ \forall_\alpha$ and $U_\alpha \neq X$ foratleastonα

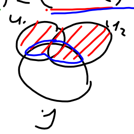
$\Rightarrow U_\alpha \in \mathcal{T} \Rightarrow U_\alpha$ isopenonX

$\Rightarrow \bigcup_{\alpha} U_\alpha$ open X

$\bigcup_{\alpha} (U_\alpha \cap Y) = \left(\bigcup_{\alpha} U_\alpha\right) \cap Y$

$(U_1 \cap Y) \cup (U_2 \cap Y) = (U_1 \cup U_2) \cap Y$





$\Rightarrow$ open

Basis

If $\mathcal{B}$ is a basis for the topology on X

then the collection

$$\mathcal{B}_Y = \{B \cap Y \mid B \in \mathcal{B}\}$$

is a basis for the subspace topology.

Given U is open on X and $y \in U \cap Y$

$y \in U \cap Y \Rightarrow y \in U$ and $y \in Y$

$U \subseteq X$ and $\mathcal{B}$ on X.

$\Rightarrow \exists\ B \in \mathcal{B}$ s.t. $y \in B$ and $B \subseteq U$

$\Rightarrow$ Since $y \in U \cap Y$ and $y \in B$

$\Rightarrow y \in B \cap Y \subseteq U \cap Y$

If Y is a subspace of X then a set U is open in Y (or open relative to Y) if it belongs to the topology on Y and U is open on X if it belongs to the topology of X.

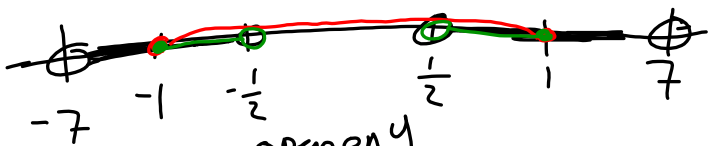
$X = \mathbb{R}$ $\quad Y = \mathbb{Z}$

$(\frac{1}{4}, \frac{1}{2}) \in \mathcal{T}_X$ $\quad (\frac{1}{4}, \frac{1}{2}) \notin \mathcal{T}_Y$

$X = \mathbb{R}$ $\quad Y \in [-1, 1]$ $\quad B = \{x \mid \frac{1}{2} < |x| \leq 1\}$

$[-1, -\frac{1}{2}) \cup (\frac{1}{2}, 1]$

$B = Y \cap ((-7, -\frac{1}{2}) \cup (\frac{1}{2}, 7))$



open on Y

Not open on $\mathbb{R}$

$\exists$ no open set $(a,b) \in \mathcal{T}_X$

s.t. $-1 \in (a,b) \subseteq [-1, -\frac{1}{2}) \cup (\frac{1}{2}, 1]$

Lemma:

Let Y be a subspace of X. If U is open in Y and Y is open in X, then U is open in X.

Given U is open in Y
$\Rightarrow U = Y \cap V$ for some $V \in X$
$\Rightarrow$ Since V and Y are open in X
then $Y \cap V$ is open in X
Since $U = Y \cap V$
$\Rightarrow U$ is open X

Thm 16.3

If A is a subspace of X and B is a subspace of Y, then the product topology on AxB is the same as the topology AxB inherits as a subspace of XxY.

Let U be a basis element for X
V be a basis element on Y
$\Rightarrow U \times V$ is the basis element on product topology on $X \times Y$
$(U \times V) \cap (A \times B)$ would be basis element for subspace topology on $A \times B$

$$(U \times V) \cap (A \times B) = (U \cap A) \times (V \cap B)$$

$U \cap A$ is open on A and $V \cap B$ is open on B
Basis element for product topology on $A \times B$
$(U \cap A) \times (V \cap B)$