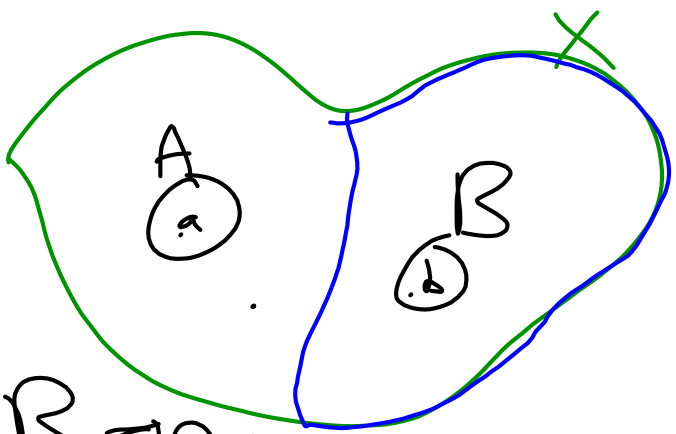
Munkres 23.3

Connected Spaces

Def. A separation of X is a pair of U, V disjoint non-empty sets, whose union is X. The space is connected if a separation does not occur.



1) A, B open
2) $A \cup B = X$
3) $A \cap B = \emptyset$
4) $A \neq \emptyset$ $B \neq \emptyset$

Claim:

A space X is connected if and only if the only subsets that are both open + closed are the $\emptyset$ and X.

A = A close

$Y - B = A$

Proof:

Suppose $\exists U$ s.t. $U \neq \emptyset$, $U \neq X$ and is both open + closed on X.

Let $V = X - U$ V is nonempty

V = open

$V \cap U = \emptyset$ (U open + open)(V closed + closed)

$\Rightarrow$ a separation occurs.

X disconnected

Lemma 23.1

If Y is a subspace of X, a separation is pair of disjoint nonempty A & B whose union is Y, neither of which contains a limit point of the other. The space Y is connected if there does not exist a separation.

Proof: Given $A, B \neq \emptyset$

$A \cup B = Y$ $A \cap B = \emptyset$

Let this be a separation.

A is both open and closed on Y. $\bar{A} \cap Y = A$ $Y - B = A \Rightarrow$ closed.

since $\bar{A} \cap Y = \emptyset$ $B \cap A = \emptyset$

$A = \bar{A} \cap Y \Rightarrow B \cap (\bar{A} \cap Y) = \emptyset$

$\Leftarrow B$ Similarly

$\bar{A}$ is union of A and its limit points by Def.

B does not contain limit points of A because assumed a separation.

(closuer $A) = \overline{A}$

$A \subset X$

$(\overline{A} - A) \cup A$

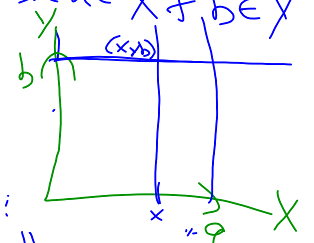
limit points of A

~~Lemma~~ Theorem 23.3

The union of connected subspaces of X that have a point in common are connected.

Lemma 23.6
A finite cartesian product of connected spaces is connected.

Proof: Choose a base pair (a,b) in $X \times Y$ s.t. $a \in X$ + $b \in Y$



Homeomorphic

$$T_x = (X \times b) \cup (x \times Y)$$

$$\bigcup_{x \in X} T_x = X \times Y$$

thm 23.5
proves that the union of two nonempty sets = topology