Compact Surfaces

Def: A collection $\mathcal{A}$ of subsets of a space X is said to cover X if the union of the elements $\mathcal{A}$ is equal to X.

An open covering of X is when the elements are open subsets of X.

Def: A space X is said to be compact if every open covering $\mathcal{A}$ of X contains a finite subcollection that also covers X.

Ex: Any space X containing finitely many points is compact

Let $\mathcal{A}$ be an open covering of X

Where $X = \{x_1, x_2, \ldots, x_n\}$

$x_i \in A_i$, s.t. for some i, $A_i \in \mathcal{A}$

$X = \bigcup_{i=1}^{n} A_i \Rightarrow \exists$ a finite subcover

Therefore, X is compact

Thm 26.2 Every closed subset of a compact space is compact

Proof: Let X be a compact space

Let A be a closed subset of X

Let $\mathcal{B}$ be an open cover of A

Since it's given that A is closed, $X-A$ is open

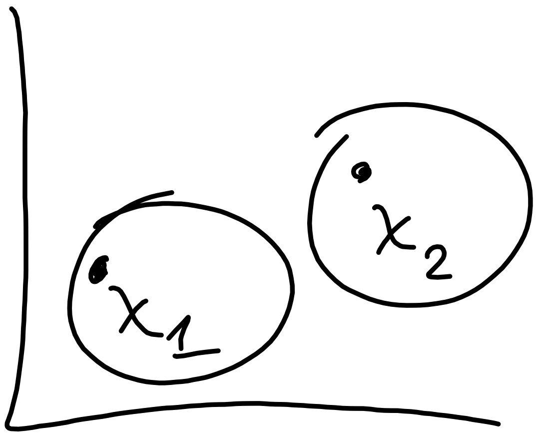
$\Rightarrow \mathcal{B} \cup (X-A)$ is an open cover of X

Because the finite subcover covers X, it also covers A

Therefore, $\exists$ a finite subcover of $\mathcal{B}$ that covers A

Hence, A is compact $\square$

Def: A topological space X is called a Hausdorff space if for each pair x_1, x_2 of distinct points on X, $\exists$ neighborhoods U_1 and U_2 of x_1 and x_2 that are disjoint



Thm 26.3 Every compact subset of a Hausdorff space is closed

Proof: Let X be a Hausdorff space

Let Y be a compact subset of X

Let $x \in X-Y$

For each point $y \in Y$, $\exists$ disjoint neighborhoods U_y and V_y of points x and y

$\{V_y \mid y \in Y\}$ is a covering of Y by sets open in X

$\Rightarrow \exists$ a finite subcover, $Y = V_{y_1} \cup V_{y_2} \cup \ldots \cup V_{y_n}$, that corresponds to points $y_1, y_2, \ldots, y_n$ in Y

Let $U = U_{y_1} \cap U_{y_2} \cap \ldots \cap U_{y_n}$

$x \in U$ and $x \in U_{y_i}$ $\forall$ $i = 1, 2, \ldots, n$

U is open since it's a finite intersection of open sets

$U_{y_i} \cap V_{y_i} = \emptyset$

$U \subset U_{y_i} \Rightarrow U \cap V_{y_i} = \emptyset$ $\forall i$

$\Rightarrow U \cap (V_{y_1} \cup V_{y_2} \cup \ldots \cup V_{y_n}) = \emptyset$

$\Rightarrow U \cap Y = \emptyset$

$\Rightarrow U$ is an open set in $X-Y$ containing x

Hence, $X-Y$ is open

Therefore, Y is closed $\square$