Definition:
If f and f' are continuous maps of the space X into the space Y, we say that f is **homotopic** to f' and there exists a continuous map F: X x I -> Y such that
F(x,0) = f(x) and F(x,1) = f'(x) for each x in X.

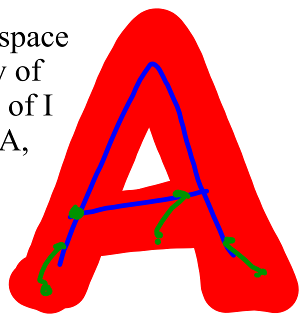
t $\quad I=[0,1]$

The map F is called a **homotopy** between f and f'.
If f is homotopic to f', we write

$$f \simeq f'$$

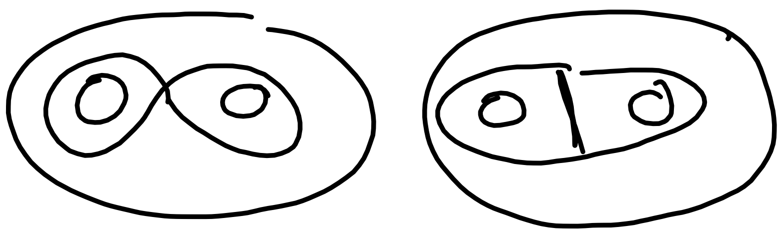
Deformation Retracts:

the deformation retraction of a space X onto a subspace A is a family of maps f_t: X -> X, t is an element of I s.t. f_0= the identity map, $f_1(x)$= A, and $f_t|A$=identity map for all t



Homotopy Equivalence:

two spaces are considered to be homotopy equivalent in general if the two spaces X and Y have a space Z containing both X and Y as deformation retracts



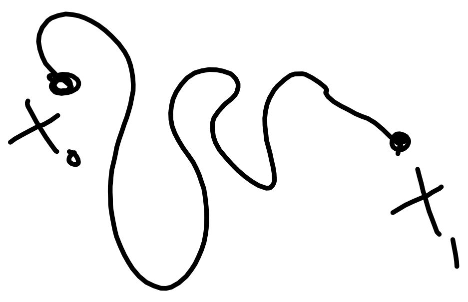
A B C D
E F G H I
J K L M
N O P Q
R S T U
V W X Y

Which letters are homotopically equivalent?

PATH

Let X be a topological space. A path from x_0 to x_1 in X is a continuous function $f: \underline{I} \to X$ s.t.

$f(0) = x_0$ $f(1) = x_1$

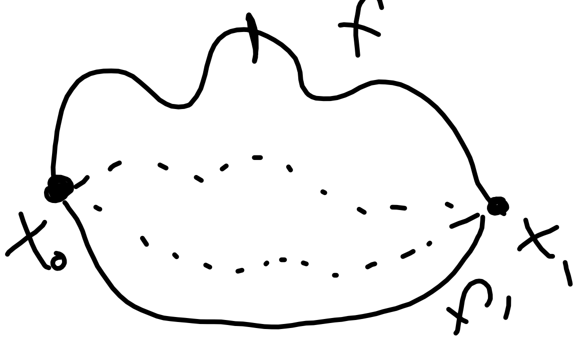


Definition:

If f and f' are two paths mapping the interval I in the topological space X, they are said to be path homotopic if they have the same x_0 and x_1, and if there is a continuous map F: I x I -> X such that

F(s,0) = f(s) and F(s,1) = f'(s),
F(0,t) = x_0 and F(1,t) = x_1,
for each s in I and each t in I.

position time



3 Properties of Equivalence Relations of Paths:

Reflexivity: a=a

$f \simeq f$

Symmetry: a=b b=a

$f \simeq f'$ $f' \simeq f$

Transitivity:

a=b b=c a=c

$f \simeq f'$ $f' \simeq f''$ $f \simeq f''$

PROOF

Reflexivity

Given f, it is trivial that $f \simeq f$

$F(x,t) = f(x)$

Symmetry:

Given $f \simeq f'$

Let F be a homotopy between f & f'

Then $G(x,t) = F(x, 1-t)$

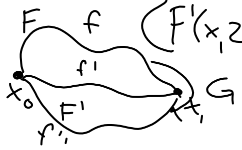
Transitivity

Let $f \simeq f'$ & $f' \simeq f''$

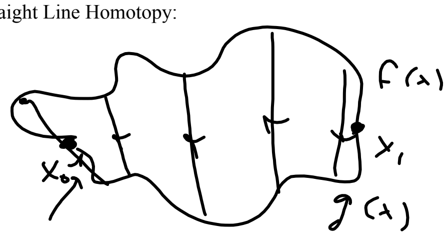
Let F be a homotopy b/w f & f'

$G: X \times I \to Y$ Let F' be from f' to f''

$$G(x,t) = \begin{cases} F(x, 2t) & t \in [0, 1/2] \\ F'(x, 2t-1) & t \in [1/2, 1] \end{cases}$$



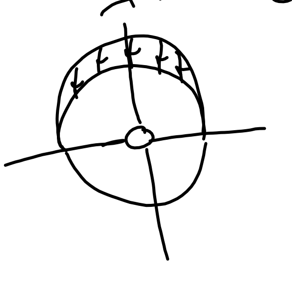
Straight Line Homotopy:



Convexity:

Let A be a convex subspace of $\mathbb{R}^n$ this means that for any 2 points a, b of A, the straight line segment joining a & b is contained in A for the straight line homotopy F b/w them has an image set in A.

$X = \mathbb{R}^2 - \{0\}$



$f(s) = (\cos \pi s, \sin \pi s)$

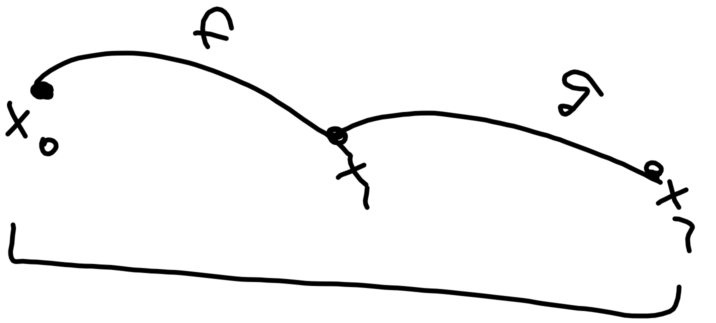
$g(s) = (\cos \pi s, 2 \sin \pi s)$

$h(s) = (\cos \pi s, -\sin \pi s)$

Definition:
If f is a path in X from x_0 to x_1, and g is a path from x_1 to x_2, we define the product of f*g to be the path h given by the equations

$$h(s) = \begin{cases} f(2s) & \text{for } s \text{ in } [0, 1/2] \\ g(2s-1) & \text{for } s \text{ in } [1/2, 1] \end{cases}$$

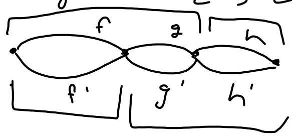
The function h is well defined and continuous. It is a path in X from x_0 to x_2. We think of h as the path whose first half is the path f and whose second half is the path g.



Operation * has following properties:

Associativity

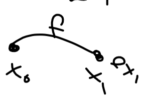
$[f*g]*[h] = [f]*[g*h]$



Right + Left Identities

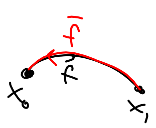
Given $x \in X$, let e_x denote the constant path $e_x: I \to X$ carrying all of I to the point x. If f is from x_0 to x_1

$[f]*[e_{x_1}] = [f]$ $[e_{x_0}]*[f] = [f]$



Inverse

Given the path in X from x_0 to x_1, let $\bar{f}$ be the path defined by $\bar{f}(s) = f(1-s)$



$[f]*[\bar{f}] = [e_{x_0}]$

$[\bar{f}]*[f] = [e_{x_1}]$