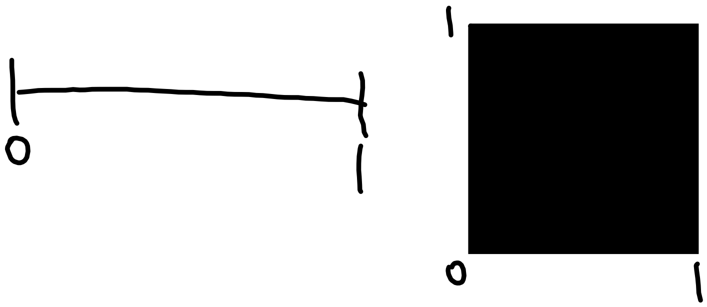
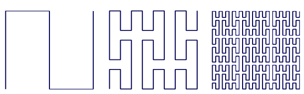
Section 44: Space Filling Curves

Def: There exists a continuous map f from the unit interval, I = [0,1], to the unit square, I^2, whose image fills the area of I^2. This image is known as a space filling curve.

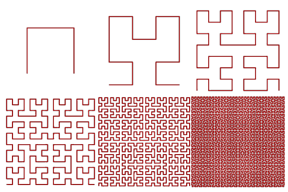


Examples of Space Filling Curves

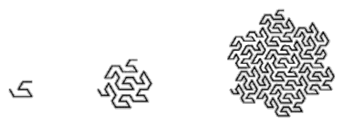
Peano Curve



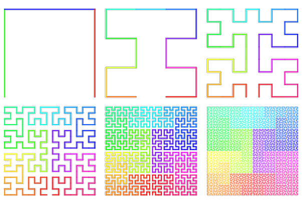
Hilbert Curve



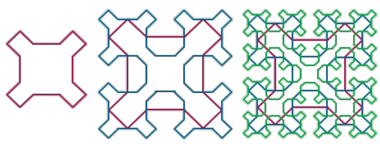
Gosper Curve



Moore Curve

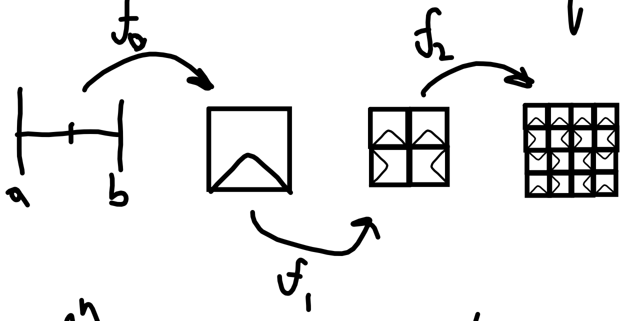


Sierpiński Curve



Theorem 44.1: Let I be the unit interval [0,1]. There exists a continuous map f: I → I^2 whose image fills up the area of the square I^2.

Step 1: f is the limit of a sequence of functions f_n.



4^n squares each of $\frac{1}{2^n}$ side length.

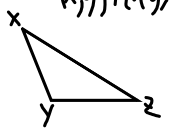
Theorem 44.1: Let I be the unit interval [0,1]. There exists a continuous map f: I → I^2 whose image fills up the area of the square I^2.

Step 2: Let $d(x,y)$ denote the square metric $\mathbb{R}^2$

$d(x,y) = \max\{|x_1-y_1|, |x_2-y_2|\}$

Def: A metric d: X X → R

1) $d(x,y) \geq 0$
2) $d(x,y) = 0$ iff $x = y$
3) $d(x,y) = d(y,x)$
4) $d(x,y) + d(y,z) \geq d(x,z)$



Def: A sup. of set $S \subset T$ is the smallest element $x \in T$ s.t. $x > s \in S$ $\forall s$ LUB

$\rho(f,g) = \sup\{d(f(t), g(t)) \mid t \in I\}$

Since I^2 is closed on $\mathbb{R}^2$ it is complete in the square metric.

Def: A complete space contain all LUB.

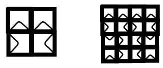
$\mathcal{C}(I, I^2)$ = the topological space $I \times I^2$ together w/ the metric topology

complete w/ metric under ρ, we can prove f_n is a CAUCHY SEQUENCE!

Let (X,d) be a metric space. A sequence X_n of points of X is a Cauchy Sequence if for any $\epsilon>0$,

$\exists$ some integer N s.t. $d(X_n, X_m) < \epsilon$ for $n,m \geq N$.

f_n to f_{n+1} 4^n squares to 4^{n+1} squares, and the side lenth of the squares reduces from $1/2^n$ to $1/2^{n+1}$.



$\rho(f_n, f_{n+1}) \leq 1/2^n$

$\forall m>n, \rho(f_n,f_m) \leq 1/2^n + 1/2^{n+1} + \dots + 1/2^{m-1} < 2/2^n$

$\sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^i = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2$ $2/2^n$ is a sequence describing the distance between a point $f_n(t)$ and $f_{n+1}(t)$

MATHMAGIC in complete spaces, Cauchy sequences converge!

Theorem 44.1: Let I be the unit interval [0,1]. There exists a continuous map f: I → I^2 whose image fills up the area of the square I^2.

Step 3: Because $C^0(I,I^2)$ is complete, the seqence f_n converges to a continuous function $f: I \to I^2$. We want to prove that f is surjective.

Let $x \in I^2$. We want to show that $x \in f(I)$.

$f_n(t)$ is at most $1/2^n$ distance from x. Given $\epsilon>0$,

Choose some N s.t. $\rho(f_N,f) < \epsilon/2$ and $1/2^N < \epsilon/2$.

$\exists$ some $t_0 \in I$ s.t. $d(x, f_N(t_0)) < 1/2^N$.

Since $d(f_N(t), f(t)) \leq \epsilon/2$

$$d(x, f(t_0)) \leq d(x, f_N(t_0)) + d(f_N(t_0), f(t_0)) < \epsilon/2 + \epsilon/2 \leq \epsilon$$

$\Rightarrow$ ϵ neighborhood of x intercepts $f(I)$.

$\Rightarrow$ x belongs to the closure of $f(I)$. Since I is compact, $f(I)$ is compact and therefore closed.

$\Rightarrow$ $f(I)$ contains all its limit points

$\Rightarrow$ $x \in f(I)$.

$\Rightarrow$ f is surjective, or onto.

Hence, a one-dimensional space can be mapped to a two dimensional space $\square$