The Fundamental Group

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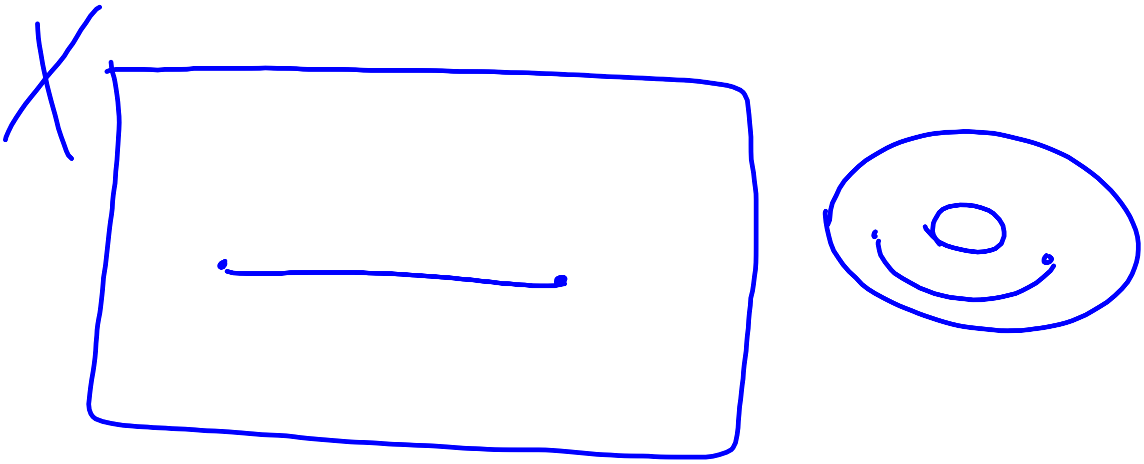
Def: Let X and Y be topological spaces; let f: $X \rightarrow Y$ be a bijection. If both the function and the inverse function, f^{-1} : $Y \rightarrow X$, are continuous then f is called a *homeomorphism.*

Def: A *homomorphism* $f : G \rightarrow G'$ is a map such that

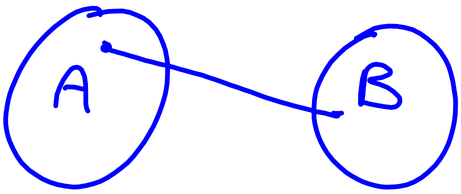
1. $f(x, y) = f(x) \cdot f(y)$ for all x, y
2. $f(e) = e'$, where e and e' are identities of G and G', respectively
3. $f(x^{-1}) = f(x)^{-1}$, where the exponent -1 denotes the inverse

Def: For groups, $f : G \rightarrow H$ is an *isomorphism* if f is bijective and a homomorphism

Def: A topological space X is *path connected* if for any two points in X we can find a path which connects them and entirely lies in X

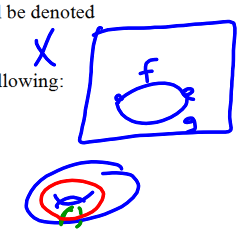


$X = A \cup B$



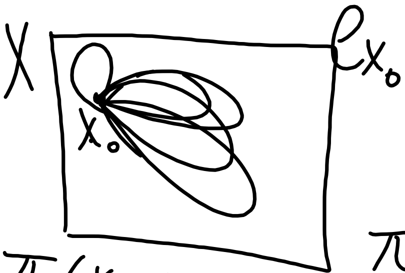
Def: the equivalence class of a path f under the equivalence class of homotopy will be denoted $[f]$ and is called the *homotopy class* of f.

Def: A group $(G, *)$ is a set with a binary operation $*$: $G \times G \rightarrow G$ satisfying the following:

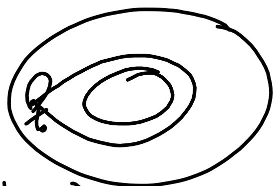


1. Identity: there exists an e in G such that $g*e = e*g = g$ for all g in G
2. Associativity: $g*(h*k) = (g*h)*k$
3. Inverse: for g in G, there exists a g^{-1} in G such that $g*g^{-1} = e = g^{-1}*g$

Def: Let X be a space; let x_0 be a point of X. A path in X that begins and ends at x_0 is called a ***loop*** based at x_0. The set of path homotopy classes of loops based at x_0, with the operation $*$, is called the ***fundamental group*** of X relative to the ***base point*** x_0. It is denoted by $\pi_1(X, x_0)$.



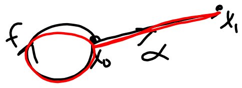
$\pi_1(X, x_0) = 0$



$\pi_1(X, x_0) = 1$

2

Def: Let α be a path in X from x_0 to x_1. We define the map
$\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1)$
by the equation
$\hat{\alpha}([f]) = [\bar{\alpha}] * [f] * [\alpha]$



Theorem 52.1: The map $\hat{\alpha}$ is a group isomorphism

Corollary 52.2: If X is path connected and x_0 and x_1 are two points of X, then $\pi_1(X, x_0)$ is isomorphic to $\pi_1(X, x_1)$.

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Def: A *homomorphism* $f : G \to G'$ is a map such that

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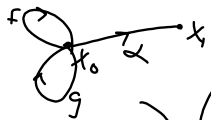
Def: For groups, $f : G \to H$ is an *isomorphism* if f is bijective and a homomorphism

Proof: Step 1 - homomorphism

· show $\hat{\alpha}([f]*[g]) = \hat{\alpha}([f]) * \hat{\alpha}([g])$

$\hat{\alpha}([f]*[g]) = [\bar{\alpha}]*[f]*[g]*[\alpha]$

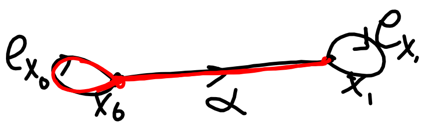


$= ([\bar{\alpha}]*[f]*[\alpha])*([\bar{\alpha}]*[g]*[\alpha])$

$= \hat{\alpha}([f]) * \hat{\alpha}([g])$

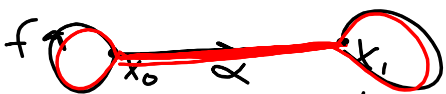
· e_{x_0} and e_{x_1} are the identities of $\pi_1(X, x_0)$ and $\pi(X, x_1)$, respectively

$\hat{\alpha}[e_{x_0}] = [\bar{\alpha}]*[e_{x_0}]*[\alpha] = [e_{x_1}]$



· Show $\hat{\alpha}([f^{-1}]) = \hat{\alpha}([f])^{-1}$

$\hat{\alpha}([f^{-1}]) = [\bar{\alpha}] * \overset{([\bar{f}])}{[f^{-1}]} * [\alpha]$



$= ([\bar{\alpha}]*[f]*[\alpha])^{-1} = \hat{\alpha}([f])^{-1}$

Homomorphism

Step 2 - bijective ⇒ inverse

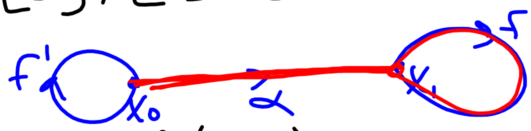
$f(f^{-1}(x)) = f^{-1}(f(x)) = x$

Let $\hat{\beta}: \pi_1(X, x_1) \to \pi_1(X, x_0)$

$\hat{\beta} = [\alpha] * [f] * [\bar{\alpha}]$

$\hat{\alpha}(\hat{\beta}[f]) = [\bar{\alpha}] * [\hat{\beta}[f]] * [\alpha]$

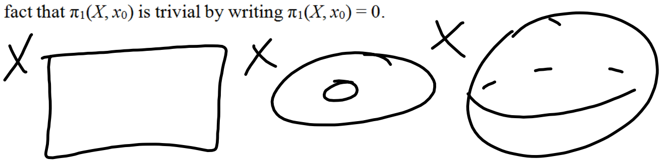
$= [\bar{\alpha}] * [\alpha] * [f] * [\bar{\alpha}] * [\alpha] = [f]$



Simarly $\hat{\beta}(\hat{\alpha}[f]) = [f]$

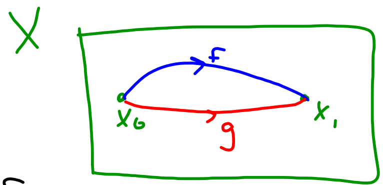
⇒ Hence, $\hat{\alpha}: \pi_1(X, x_0) \to \pi_1(X, x_1)$ is a group isomorphism □

Def: A space X is said to be *simply connected* if it is a path-connected space and if $\pi_1(X, x_0)$ is the trivial (one-element) group for some $x_0 \in X$, and hence for every $x_0 \in X$. We often express the fact that $\pi_1(X, x_0)$ is trivial by writing $\pi_1(X, x_0) = 0$.



Lemma 52.3: In a simply connected space X, any two ~~points~~ paths having the same initial and final points are path homotopic.

Proof: Let f and g be two paths in X w/ initial point x_0 and final point x_1



$f * \bar{g}$ is a loop based at x_0. Since X is simply connected, $\pi_1(X) = 0$ and all loops are homotopic to the constant loop e_{x_0}.

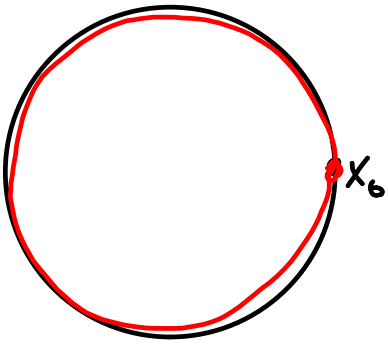
$[f * \bar{g}] * [g] \simeq [e_{x_0}] * [g] \simeq [g]$

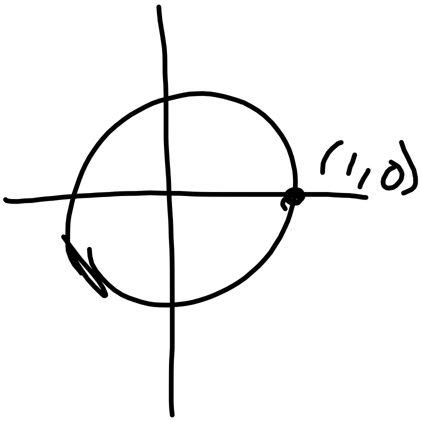
$[g] * [\bar{g}] \simeq e_{x_0}$

$[f] \simeq [g]$ □

The Fundamental Group of Circle

$$S^1 = \{(x,y) \mid x^2+y^2=1\}$$
$$= \{(\cos\theta, \sin\theta) \mid 0 \le \theta \le 2\pi\}$$





Theorem: $\pi_1(S^1, x_0)$ is ~~homotopic~~ to the ~~set~~ group of all integers.

homomorphic

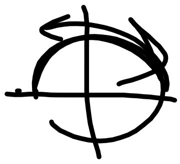
$$\pi_1(S^1, x_0) \cong \mathbb{Z}$$

$$\phi : \mathbb{Z} \to \pi_1(S^1, \underset{(1,0)}{x_0})$$

defined by $\phi(n) = [w_n]$

$$w_n(s) = (\cos 2\pi ns, \sin 2\pi ns)$$

For $n > 0$



For $n < 0$

For $n = 0$ $e_{(1,0)}$