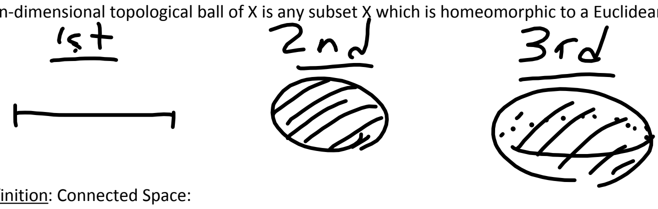
Brouwer Fixed-Point Theorem

Theorem: The continuous function of a ball of any dimension to itself must have a fixed point.

$$f: B^n \to B^n \ \exists x \in B^n, \text{ s.t. } f(x) = x$$

Definition: Ball

An n-dimensional topological ball of X is any subset X which is homeomorphic to a Euclidean n-ball.



Definition: Connected Space:

A topological space that cannot be represented by a union of two or more disjoint nonempty open sets.

Definition: Continuity

A function f: X->Y is continuous if for every set $U \subseteq Y$, the preimage $f^{-1}(U)$ is open in X.

Definition: Fixed-Point:

A fixed point of a function is an element of the function's domain that maps to itself by the function.

A fixed point of a function f is the point x such that f(x) = x.

Proofs

First Dimension

Let B^1 be our ball of the first dimension

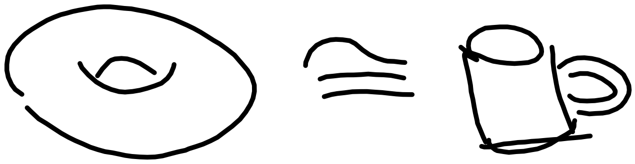
let f be out continuous function so that $f:B^1 \to B^1$

Recall that the n-ball of dimension one is an interval [x,y]

Through homeomorphism, the ball, or interval, can be compressed into the unit interval [0,1].

Definition: Homeomorphism:

A continuous function that maps on topological ~~surface~~ Space to another that also has a continuous inverse function.



First Dimension

B^1 is our ball of the first dimension that is equal to the unit interval [0,1].

The function f is defined by f: $B^1 \to B^1$ and f is continuous.

Proof by contradiction

Let us assume that there is not an element $x \in B^1$ such that $f(x) = x$,

$\forall x \in B^1 \quad f(x) \neq x \qquad B^1 = [0,1]$

$f(x) > x$

$f(x) < x$

$$B^1 = \underbrace{\{x \in B^1 \mid f(x) > x\}}_{A} \cup \underbrace{\{x \in B^1 \mid f(x) < x\}}_{B}$$

$f(1) < 1 \in B$

$f(0) > 0 \in A$

Definition(§23 Connected Spaces): Separation of a space: If X is a topological space, A separation of X is two sets U and V of disjoint nonempty open subsets of X whose union is X. X is connected if a separation does not exist.

1. Disjoint (Intersection of A and B is equal to the empty set)

$A \cap B = \emptyset$

2. Nonempty

$1 \in B$

$0 \in A$

3. Open

Definition (Wolfram Mathworld): Open Set: If U is a subset of a metric space, U is open if every point in U has a neighborhood lying in the metric space.

Definition (Wikipedia): Neighborhood of a point: A set containing the point where one can move that point without leaving the set.

A function f: X->Y is continuous if for every set $U \subseteq Y$, the preimage $f^{-1}(U)$ is open in X.

$f: B^1 \to B^1$

A and B are open in B^1

4. The union of A and B is equal to B^1

$B^1 = A \cup B$

Definition (§23 Connected Spaces): Connected space: A topological space that cannot be represented as the union of two or more disjoint nonempty open subsets. A space is connected if there does not exist a separation.

(§24 Connected Subspaces of the Real Line) Example 3: The unit ball ([0,1]) is path connected.

(Wikipedia) Path Connectedness: Every path-connected space is connected, however, not every connected space is path-connected.

B^1 is connected

CONTRADICTION

Hence, $\exists\, x \in B^1$ s.t. $f(x) = x$ and BFPT is true in the first dimension.

Second Dimension

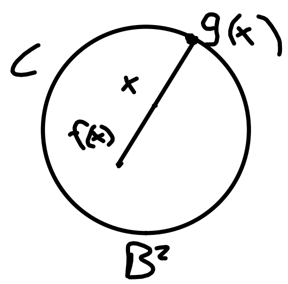
Let B^2 be our ball of the second dimension

let f be the continuous function so that f: $B^2 \to B^2$

We must again prove that there exists $x \in B^2$ such that $f(x) = x$

Proof by contradiction

We will prove this through contradiction by assuming that there is no point $x \in B^2$ such that $f(x) = x$

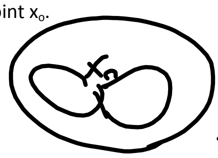


$g : B^2 \to C$

Definition (Wikipedia): Retraction: A continuous mapping from the entire space into a subspace that preserves the position of all points in that subspace. Let X be a topological space and A be a subspace of X. A continuous map r from X to A is a retraction if the map is the identity map on A, or r(a) = a for all a in A.

g is continuous

Definition (§52 The Fundamental Group): Fundamental Group: Let X be a space and let x_o be a point on X. A path that begins and ends at x_o is a looped based on x_o. The set of path homotopy classes of loops is called the fundamental group of X relative to the base point x_o.



$\pi_1(X, x_o): 0$

Definition (§52 The Fundamental Group): Simply Connected Space: A space X is simply connected if it is a path-connected space and if $\pi_1(X, x_o)$ is the trivial group for some x_o X. $\pi_1(X, x_o)$ is the trivial group if $\pi_1(X, x_o) = 0$.

B^2

Fundamental Group ; trivialgroup: 0

Definition (Wikipedia): Homotopy Group: A group of homotopy classes which are base point preserving maps from an n-dimensional sphere into a given space.

Definition (Wikipedia): Infinite Cyclic Fundamental Group: The homotopy class of a circle consists of all the loops which wind around the circle a given number of times. The fundamental group of a circle is the additive group of real integers (Z).

C

Fundamental Group: $\mathbb{Z}$

Definition (§52 The Fundamental Group): Induced homomorphism: Let h: $(X, x_o) \to (Y, y_o)$ be a continuous map. $h_*: \pi_1(X, x_o) \to \pi_1(Y, y_o)$ by the equation $h_*([f]) = [h \circ f]$. The map h_* is called the homomorphism induced by h, relative to the base point x_o.

g is continuous

$\exists\ g*$

Corollary 52.2: If X is path connected and x_0 and x_1 are two points of X, then $\pi_1(X, x_o)$ is isomorphic to $\pi_1(X, x_1)$.

Definition (§52 The Fundamental Group): Isomorphism: the homomorphism f is called an isomorphism if it is bijective, which means it is both surjective and injective.

$\pi_1(B^2)$ is isomorphic to $\pi_1(C)$

$g: B^2 \to C$

$g*: \pi_1(B^2) \to \pi_1(C)$

We will denote the point p = (1,0) as the base point for both C and B^2

Definition (Wikipedia): Inclusion map: If A is a subset of B, the inclusion map i is the function that sends each element x of A to x, treated as an element of B.

$C \subseteq B^2$

$i: C \to B^2$

$g: B^2 \to C \quad g(x) = x$ for all points of C

$g_*: \pi_1(B^2) \to \pi_1(C) \quad \pi_1(B^2) \xrightarrow{g_*} \pi_1(C)$

$i_*: \pi_1(C) \to \pi_1(B^2)$

$$\pi_1(C) \xrightarrow{i_*} \pi_1(B^2) \xrightarrow{g_*} \pi_1(C)$$

Definition (Wikipedia): Identity Map: A function that always returns the same value that was used in its argument. The function f is an identity map is f(x) = x for all values of x.

$g(x) = x$

$i(x) = x$

$g \circ i(x) = x$

Theorem 52.4: If h:(X, x_0) -> (Y, y_0) and k:(Y, y_0) -> (Z, z_0) are continuous, then $(k \circ h)_* = k_* \circ h_*$. If i: (X, x_0) -> (X, x_0) is the identity map, then i_* is the identity homomorphism.

$g_* \circ i_*$

(§3 Relations) Question 4 part e) If $g \circ f$ is surjective, g is surjective, but f is not necessarily surjective.

Definition (§2 Functions): Surjective (onto): for the function f: A -> B, every element B is the image of some element of A under the function f.

g_* is surjective or onto

$\pi_1(B^2)$ is trivial group

$\pi_1(C)$ is $\mathbb{Z}$

$g_*: \pi_1(B^2) \to \pi_1(C)$

Hence $\exists\, x \in B^2$ s.t. $f(x) = x$