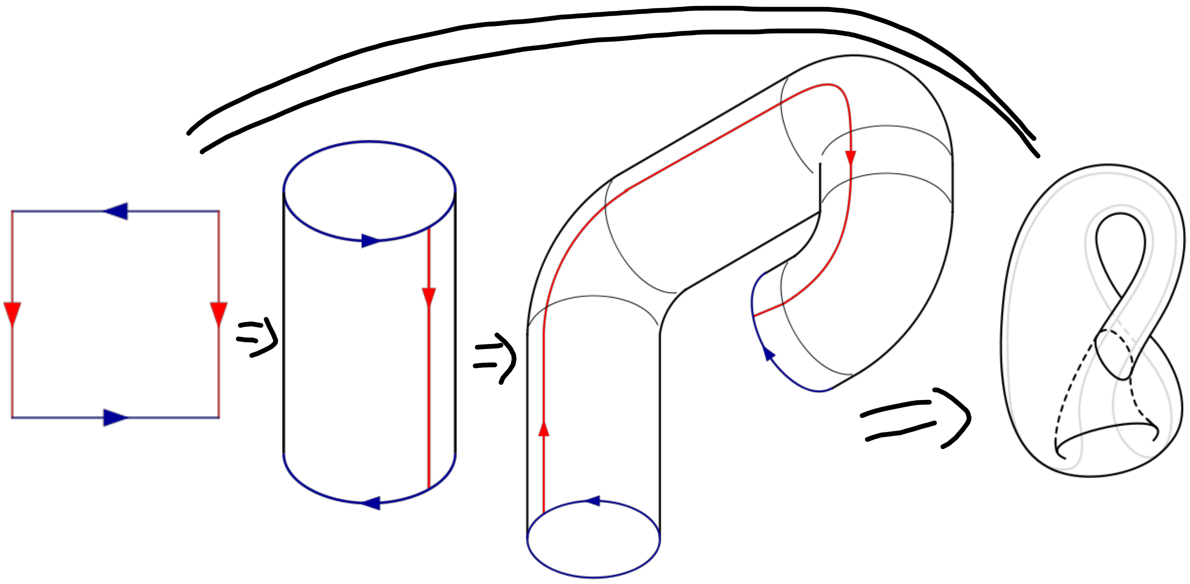
Quotient Topology

Many topologists think of quotient topology as the result of identifying and "gluing together" points in a topological space.

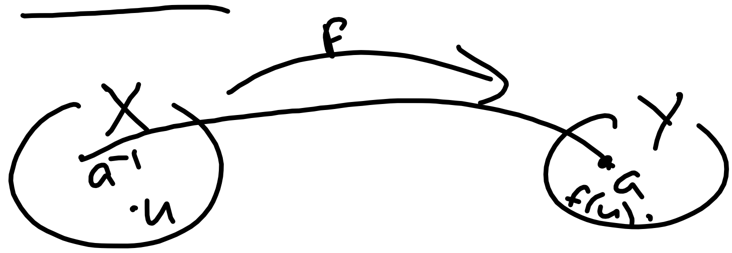


Definition: Quotient Map

Let X and Y be topological spaces; let p: X → Y be a surjective map. The map p is said to be a quotient map provided a subset of U of Y is open in Y if and only if *p-1*(U) is open in X.

To simplify, a quotient map needs to have the three following qualities:

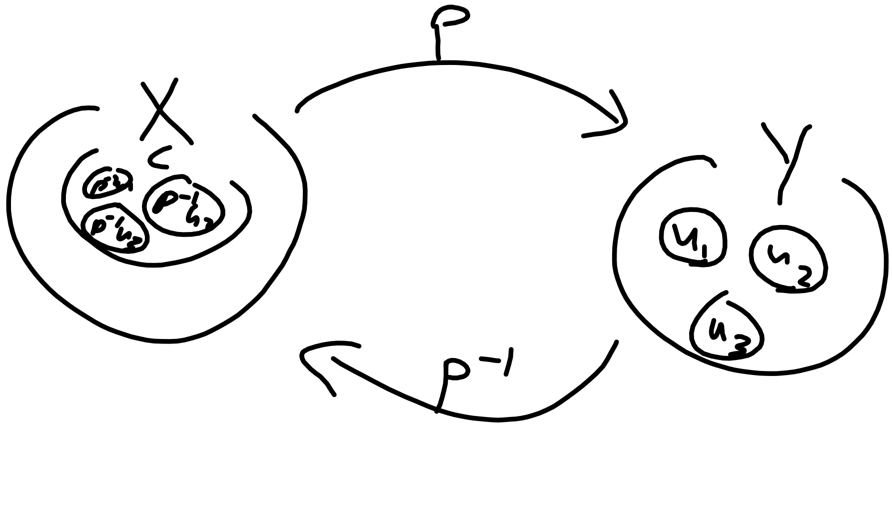
- Surjective
- Continuous
- Open or closed map



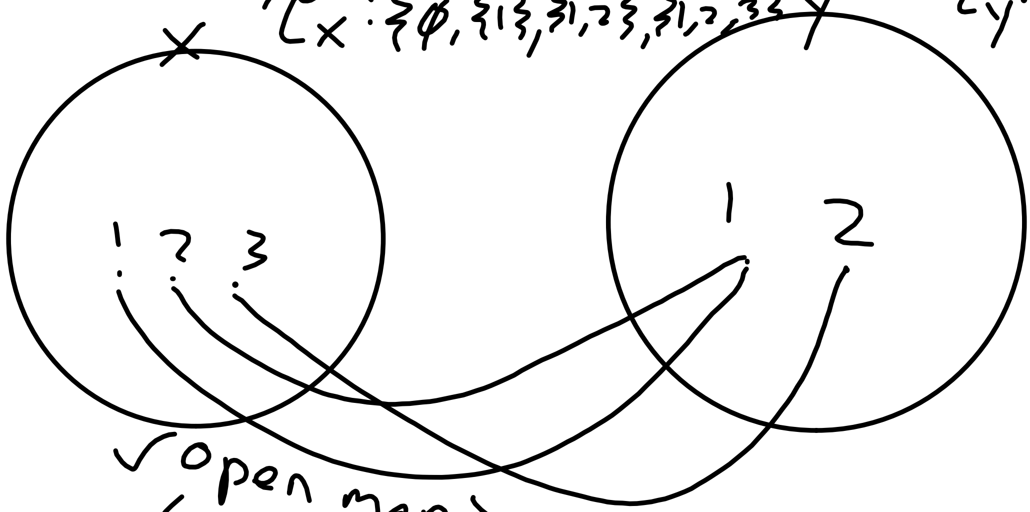
Definition: Saturated

We say that a subset C of X is saturated (with respect to the surjective map p: X → Y) if C contains every set *p-1*(*U*) that it intersects.

$p: X \to Y$



$\tau_X : \{\phi, \{1\}, \{1,2\}, \{1,2,3\}\}$ $\tau_Y : \{\phi, \{1\}, \{1,2\}\}$

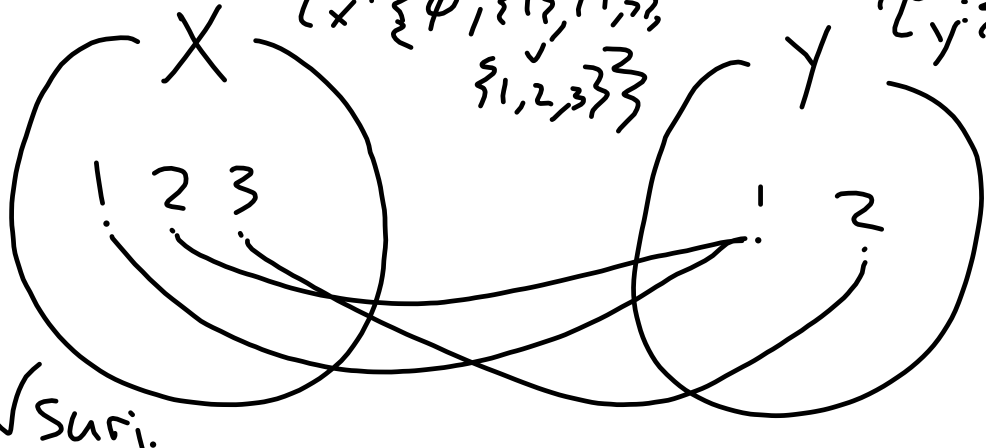


✓ open map
✓ surj.
✓ cont.
} Quotient Map

Definition: Quotient Topology

If X is a space and ~~A~~ Y is a set and if p: X → ~~A~~ Y is a surjective map, then there exists exactly one topology ~~𝒯~~ on ~~A~~ Y relative to which p is a quotient map; it is called the quotient topology induced by p.

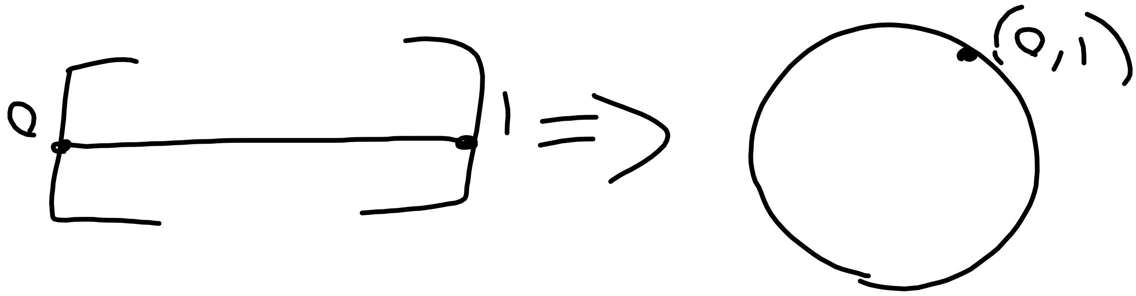
$\tau_X : \{\phi, \{1\}, \{1,2\}, \{1,2,3\}\}$ $\tau_Y : \{\phi, \{1\}, \{1,2\}\}$

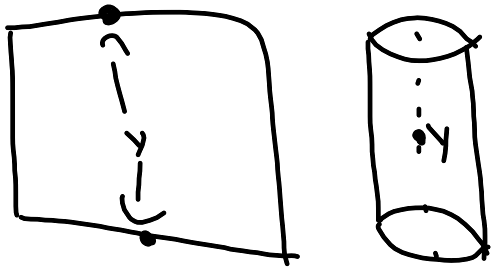


✓ surj.
✓ cont.
✓ open

Definition: Quotient Space

Let X be a topological space, and let $X^{\#}$ be a partition of X into disjoint subsets whose union is X. Let p: X $\to$ $X^{\#}$ be the surjective map that carries each point of X to the element of $X^{\#}$ containing it. In the quotient topology induced by p, the space $X^{\#}$ is called a quotient space of X.





Theorem 22.1:

Let p: X $\to$ Y be a quotient map; let A be a subspace of X that is saturated with respect to p; let q: A $\to$ p(A) be the map obtained by restricting p.

1. If A is either open or closed in X, then q is a quotient map.
2. If p is either an open map or a closed map, then q is a quotient map.

1) By subspace topology, U is open in A

$\Rightarrow U = A \cap V \Rightarrow V$ is open in X

Since $V = (U \cap A)$, an open set in subspace topology on A

Also, $p(U) \cap p(A)$ is open in subspace topology on $Y \Rightarrow$ it is open in sub. top.

Open map ✓

Since $q: \underline{A} \to p(A)$ surj., cont.

✓ Quotient map

2) $q: A \to p(A)$

Let $\overset{\text{surj.}}{U} \in p(A)$

$\underline{p^{-1}(U)} \in p^{-1}\underbrace{p(A)}_{A}$

$\overset{\text{cont.}}{p^{-1}(U)} \in p^{-1}(p(A))$ where U is open in X since p is a quotient map.

$\underline{q^{-1}(U)} \subseteq \underline{p^{-1}(U)}$

$p^{-1}(U) \in A$, which is open by subspace top. on A

$\Rightarrow p$ is an <u>open map</u>

✓ Quotient map □

Theorem 22.2:

Let p: X⟶ Y be a quotient map. Let Z be a space and let g: X⟶ Z be a map that is constant on each set *p-1(y)*, for y ∈ Y. then g induces a map f: Y⟶ Z such that f∘g=g. The induced map f is continuous if and only if g is continuous; f is a quotient map if and only if g is a quotient map.

$$\begin{tikzcd} X \arrow[d, "p"'] \arrow[dr, "g"] & \\ Y \arrow[r, dashed, "f"'] & Z \end{tikzcd}$$

Let g be cont.
To show f is cont.

Let U be an open set in Z.

$\Rightarrow g^{-1}(U)$ is an open set in X.

$\Rightarrow p(g^{-1}(U))$ is an open set in Y.

Since $f \circ g = g \Rightarrow \underline{(f \circ g)}^{-1} = g^{-1}$

$\Rightarrow p(f \circ g)^{-1}(U)$ is open in Y

#4a 21 $f: A \to B$ & $g: B \to C$
$C_0 \subset C$ $(g \circ f)^{-1}(C_0) = f^{-1}(g^{-1}(C_0))$

$p(p^{-1}(f^{-1}(U)))$ is open in Y

$\Rightarrow \underline{f^{-1}(U)}$ is open in Y

Let $\underline{f}$ be cont.
To show g is cont.

we know $\underline{f \circ p = g}$

Since p is cont. by def quotient map.

$\Rightarrow g$ is cont.

Theorem 22.2:

Let p: X Y be a quotient map. Let Z be a space and let g: X Z be a map that is constant on each set *p-1(y)*, for y Y. then g induces a map f: Y Z such that f g=g. The induced map f is continuous if and only if g is continuous; f is a quotient map if and only if g is a quotient map.

$$\begin{tikzcd} X \arrow[d, "p"'] \arrow[dr, "g"] & \\ Y \arrow[r, dashed, "f"'] & Z \end{tikzcd}$$

Let g be a quotient map
To show f is a quotient map

Let U be an open set in Z

~~f is a quotient map.~~
by similar argument to continuity proof from before we say
✓ cont.

Proving surj. of f.

we know $f \circ p = g$

p & g is quotient map

$\Rightarrow p$ & g is surj.

$\Rightarrow f$ is surj. ✓

Prove
f is an map

Let Q be an open set Y

$p^{-1}(Q)$ is open in X

$g p^{-1}(Q)$ is open in Z

$f \circ p = g$

$f(p(p^{-1}(Q)))$

$\Rightarrow f(Q)$ is open in Z

$\Rightarrow f$ is open

$\Rightarrow f$ is quotient map. □

Corollary 22.3:

Let g: X $\to$ Z be a surjective continuous map. Let $X^\#$ be the following collection of subsets of X:

$$X^\# = \{ g^{-1}(\{z\}) \mid z \in Z \}$$

Give $X^\#$ the quotient topology.

1. The map g induces bijective continuous map f: $X^\# \to$ Z, which is a homeomorphism if and only if g is a quotient map.

$$\begin{tikzcd} X \arrow[d, "p"'] \arrow[dr, "g"] & \\ X^\# \arrow[r, "f"'] & Z \end{tikzcd}$$

e.g.