$$\bigcap_{A\in\mathcal{A}} A = A_0 \cap A_1 \cap A_2 \cap \ldots \cap$$

$x \in \bigcap_{A\in\mathcal{A}} A$ if $x \in A_i \ \forall i$

$$\bigcup_{A\in\mathcal{A}} A = A_0 \cup A_1 \cup A_2 \cup \ldots$$

$x \in \bigcup_{A\in\mathcal{A}} A_i$ if $x \in A_i$ for at least one $A_i \in \mathcal{A}$

$$f\left(\bigcap_{A\in\mathcal{A}} A\right) \subseteq \bigcap_{A\in\mathcal{A}} f(A)$$

$\subseteq$: Let $y \in f\left(\bigcap_{A\in\mathcal{A}} A\right)$

$\Rightarrow f^{-1}(y) \in \left(\bigcap_{A\in\mathcal{A}} A\right)$

$\Rightarrow f^{-1}(y) \in A$ for every $A \in \mathcal{A}$

$\Rightarrow y \in f(A)$ for every $A \in \mathcal{A}$

$\Rightarrow y \in \bigcap_{A\in\mathcal{A}} f(A)$

hence $f\left(\bigcap_{A\in\mathcal{A}} A\right) \subseteq \bigcap_{A\in\mathcal{A}} f(A)$

$$f\left(\bigcap_{A\in\mathcal{A}} A\right) \subseteq \bigcap_{A\in\mathcal{A}} f(A)$$

Suppose, in addition, that f is injective (one-to-one).

To show that $f\left(\bigcap_{A\in\mathcal{A}} A\right) \supseteq \bigcap_{A\in\mathcal{A}} f(A)$

$\supseteq$: Let $y \in \bigcap_{A\in\mathcal{A}} f(A)$

$\Rightarrow y \in f(A)$ for every $A \in \mathcal{A}$

$\Rightarrow f^{-1}(y) \in A$ for every $A \in \mathcal{A}$

$\Rightarrow \exists\ x_0 \in A_0,\ x_1 \in A_1,\ x_2 \in A_2, \ldots, x_i \in A_i$

such that $f(x_i) = y$.

Since the function is injective,

$f(x_i) = f(x_j) \Rightarrow x_i = x_j = x_0$

$\Rightarrow x_0 \in A$ for every $A \in \mathcal{A}$

$\Rightarrow x_0 \in \bigcap_{A\in\mathcal{A}} A$

$\Rightarrow f(x_0) \in f\left(\bigcap_{A\in\mathcal{A}} A\right)$

Since $y = f(x_0) \Rightarrow y \in f\left(\bigcap_{A\in\mathcal{A}} A\right)$

Hence

$$f\left(\bigcap_{A\in\mathcal{A}} A\right) \supseteq \bigcap_{A\in\mathcal{A}} f(A)$$

1.2

Given $f: A \to B$ and $g: B \to C$, $A \xrightarrow{f} B \xrightarrow{g} C$

$g\circ f: A \to C$ has the rule

$\star$ $\{(a,c) \mid \text{For some } b \in B,\ f(a) = b \text{ and } g(b) = c\}$

Lemma: If a function has an inverse, then it is bijective.

f & g are inverses if $(f\circ g)(x) = x$ & $(g\circ f)(x) = x$

$$\begin{tikzcd} A \arrow[r, bend left, "f"] & B \arrow[l, bend left, "f^{-1}"] \end{tikzcd}$$

(x,x)

1.2
4. $f: A \to B$, $g: B \to C$

(a) If $C_0 \subseteq C$, show that
$$(g \circ f)^{-1}(C_0) = f^{-1}(g^{-1}(C_0)).$$

$\subseteq$: $\boxed{\text{Let } h(x) = g(f(x)) \quad h: A \to C}$ If $a_0 \in h^{-1}(C_0)$ $\Rightarrow h(a_0) \in C_0$

Let $a_0 \in (g \circ f)^{-1}(C_0)$

$\Rightarrow (g \circ f)(a_0) \in C_0$

$\Rightarrow \exists\ c_0 \in C_0$ such that $(g \circ f)(a_0) = c_0$

since f is surjective,

$\Rightarrow \exists\, b_0 \in B$ such that $f(a_0) = b_0$ and $g(b_0) = c_0$.

$\Rightarrow b_0 = g^{-1}(c_0) \Rightarrow b_0 \in g^{-1}(C_0)$

$f(a_0) = b_0 \Rightarrow a_0 = f^{-1}(b_0)$

$b_0 \in g^{-1}(C_0) \Rightarrow f^{-1}(b_0) \in f^{-1}(g^{-1}(C_0))$

Since $a_0 = f^{-1}(b_0)$, $a_0 \in f^{-1}(g^{-1}(C_0))$

and hence $(g \circ f)^{-1}(C_0) \subseteq f^{-1}(g^{-1}(C_0))$.

1.2
4. $f: A \to B$, $g: B \to C$

(a) If $C_0 \subseteq C$, show that
$$(g \circ f)^{-1}(C_0) = f^{-1}(g^{-1}(C_0)).$$

$\supseteq$:

1.2

\# 1a

2a, b, c, d

Monday, 3/21