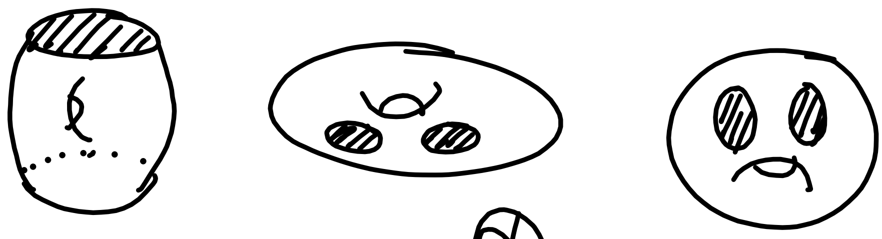
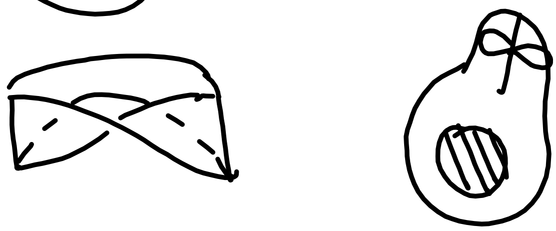
1. 2 boundary components

2. 

3. 

A **topology** on a set X is a collection $\mathcal{T}$ of subsets of X with the following properties:

1) $\emptyset, X$ are in $\mathcal{T}$
2) the union of elements in *any* (arbitrary) subcollection of $\mathcal{T}$ is in $\mathcal{T}$

$$\bigcup_{\alpha} U_\alpha \in \mathcal{T}, \qquad U_\alpha \in \mathcal{T}\ \forall \alpha$$

3) the intersection of elements of any *finite* subcollection of $\mathcal{T}$ is in $\mathcal{T}$

$$\bigcap_{i=1}^{n} U_i \in \mathcal{T}, \qquad U_i \in \mathcal{T}\ \forall i$$

A set X with a specified $\mathcal{T}$ is called a **topological space**, denoted by $(X, \mathcal{T})$.

Let $(X, \mathcal{T})$ be a topological space, and $U \subseteq X$. U is said to be **open** if $U \in \mathcal{T}$.

A set A is **finite** if it is in bijection with a finite subset of $\mathbb{Z}_+$, i.e. there exists a bijection $f: A \to \{1, 2, 3, \dots, n\}$ for some $n \in \mathbb{Z}_+$.
$\mathbb{N}$

Cor 6.7 Let $B \neq \emptyset$. The following are equivalent:
1) B is finite
2) there exists a surjection $f: \{1, 2, 3, \dots, n\} \to B$ (onto)
3) there exists an injection $f: B \to \{1, 2, 3, \dots, n\}$ (1-1)

A set is **infinite** if it is not finite. A set A is said to be **countably infinite** if there exists a bijection $f: A \to \mathbb{Z}_+$.

A is said to be **countable** if it is finite or countably infinite. Else it is said to be **uncountable**.
Note: $\emptyset$ is finite and therefore countable

Thm 7.1 Let $B \neq \emptyset$. The following are equivalent:
1) B is countable
2) there exists a surjection $f: \mathbb{Z}_+ \to B$
3) there exists an injection $f: B \to \mathbb{Z}_+$

Lemma 7.2 If C is any infinite subset of $\mathbb{Z}_+$, then C is countably infinite.

Cor 7.3 Every subset of a countable set is countable.

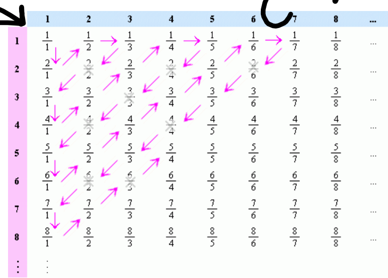
Thm 7.5 Countable union of countable sets is countable

The set of Natural Numbers is infinite and countable $\mathbb{N} = \mathbb{Z}_+$

The set of Integers is countably infinite $\to f(x) = \begin{cases} 2x, & x \geq 0 \\ 2|x|-1, & x < 0 \end{cases}$

The set of Real Numbers is uncountable

The set of Rational Numbers is countable



The set of Irrational Numbers is uncountable

Unions & intersections of finite sets are finite

Union of uncountable sets is uncountable

Union of infinite sets (countable or uncountable) is infinite

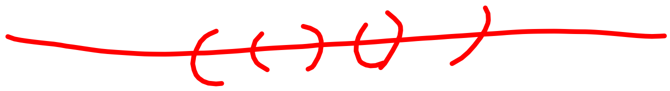
$\mathcal{T}$ is **comparable** with $\mathcal{T}'$ if $\mathcal{T}' \supseteq \mathcal{T}$ or $\mathcal{T} \supseteq \mathcal{T}'$.
If $\mathcal{T}' \supseteq \mathcal{T}$, we say that $\mathcal{T}'$ is **finer** than $\mathcal{T}$, and that $\mathcal{T}$ is **coarser** than $\mathcal{T}'$.

If X is a set, a **basis** for a topology on X is a collection $\mathfrak{B}$ of subsets of X such that:

1) for every $x \in X$, $\exists\, B \in \mathfrak{B}$ such that $x \in B$
2) Given $B_1, B_2 \in \mathfrak{B}$, if $x \in B_1 \cap B_2$, then $\exists\, B_3 \in \mathfrak{B}$ such that $x \in B_3 \subset B_1 \cap B_2$

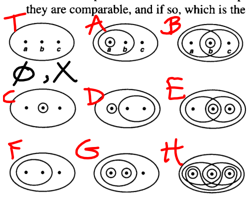
A subset U of X is said to be **open** in X (that is, to be an element of $\mathcal{T}$) if for each $x \in U$, there is a basis element $B \in \mathfrak{B}$ such that $x \in B \subset U$.

$\mathfrak{B}$ is a basis for $\mathcal{T}$ if $\mathcal{T}$ is **generated** by $\mathfrak{B}$, i.e. for any $U \subseteq X$, U open, and $x \in U$, $\exists\, B \in \mathfrak{B}$ such that $x \in B \subset U$.



2. Consider the nine topologies on the set $X = \{a, b, c\}$ indicated in Example 1 of §12. Compare them; that is, for each pair of topologies, determine whether they are comparable, and if so, which is the finer.

$\mathcal{T}$ is comparable with $\mathcal{T}'$ if $\mathcal{T}' \supseteq \mathcal{T}$ or $\mathcal{T} \supseteq \mathcal{T}'$.
If $\mathcal{T}' \supseteq \mathcal{T}$, we say that $\mathcal{T}'$ is finer than $\mathcal{T}$, and that $\mathcal{T}$ is coarser than $\mathcal{T}'$.



$A = \{\emptyset, \{a,b,c\}, \{a,b\}, \{a\}\}$

$B = \{\emptyset, \{a,b,c\}, \{a,b\}, \{b,c\}, \{b\}\}$

$C = \{\emptyset, \{a,b,c\}, \{b\}\}$

$C \subseteq B$ C is coarser than B
B is finer than C

$D = \{\emptyset, \{a,b,c\}, \{b,c\}, \{a\}\}$

$E = \{\emptyset, \{a,b,c\}, \{a,b\}, \{b,c\}, \{b\}, \{c\}\}$

$B \subseteq E$ E is finer than B
B is coarser than E

$F = \{\emptyset, \{a,b,c\}, \{a,b\}\}$

$F \subseteq A$, $F \subseteq B$, $F \subseteq E$
F is coarser than A, B, E

$G = \{\emptyset, \{a,b,c\}, \{a,b\}, \{a\}, \{b\}\}$

$G \supseteq A$, $G \supseteq C$, $G \supseteq F$
G is finer than A, C, F

$H = \{\emptyset, \{a,b,c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a\}, \{b\}, \{c\}\}$

all A thru G $\subseteq H$; H is the finest of them ALL

$T = \{\emptyset, \{a,b,c\}\}$

$T \subseteq$ all of A thru H; It is the coarsest of them all

standard topology on $\mathbb{R}$: topology generated by all open intervals $(a, b) = \{x \in \mathbb{R} | a < x < b\}$; whenever we consider $\mathbb{R}$, we assume it is given this topology unless we specifically state otherwise

1. To show $\emptyset$ & $\mathbb{R}$ are in τ