A **topology** on a set X is a collection $\mathcal{T}$ of subsets of X with the following properties:

1) $\emptyset, X$ are in $\mathcal{T}$
2) the union of elements in *any* (arbitrary) subcollection of $\mathcal{T}$ is in $\mathcal{T}$

$$\bigcup_{\alpha} U_\alpha \in \mathcal{T}, \qquad U_\alpha \in \mathcal{T}\ \forall \alpha$$

3) the intersection of elements of any *finite* subcollection of $\mathcal{T}$ is in $\mathcal{T}$

$$\bigcap_{i=1}^{n} U_i \in \mathcal{T}, \qquad U_i \in \mathcal{T}\ \forall i$$

A set X with a specified $\mathcal{T}$ is called a **topological space**, denoted by $(X, \mathcal{T})$.

Let $(X, \mathcal{T})$ be a topological space, and $U \subseteq X$. U is said to be **open** if $U \in \mathcal{T}$.

discrete topology: the collection of all subsets of X (same as power set)

indiscrete or trivial topology: the collection consisting of X and $\emptyset$ only

finite complement topology: the collection $\mathcal{T}_f$ of all subsets U of X such that $X - U$ is either finite or all of X

co-countable topology: the collection $\mathcal{T}_C$ of all subsets U of X such that $X - U$ is either countable or all of X

Let X be a topological space and $Y \subset X$. Then the collection $\mathcal{T}_Y = \{U \cap X | U \text{ open in } X\}$ is a topology on Y, called the **subspace topology**.

standard topology on $\mathbb{R}$: topology generated by all open intervals $(a, b) = \{x \in \mathbb{R} | a < x < b\}$; whenever we consider $\mathbb{R}$, we assume it is given this topology unless we specifically state otherwise

the lower-limit topology on $\mathbb{R}$: topology generated by the collection $\mathbb{R}_\ell$ all half-open intervals of the form $[a, b) = \{x \in \mathbb{R} | a \leq x < b\}$

K-topology on $\mathbb{R}$: the topology generated by the collection $\mathbb{R}_K$ of all open intervals (a, b) along with all sets of the form $(a, b) - K$, where K is the set of all numbers of the form $1/n$, for $n \in \mathbb{Z}_+$

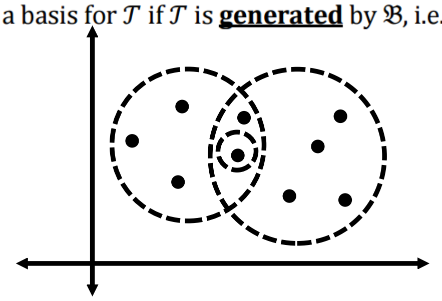
$\mathcal{T}$ is **comparable** with $\mathcal{T}'$ if $\mathcal{T}' \supseteq \mathcal{T}$ or $\mathcal{T} \supseteq \mathcal{T}'$.
If $\mathcal{T}' \supseteq \mathcal{T}$, we say that $\mathcal{T}'$ is **finer** than $\mathcal{T}$, and that $\mathcal{T}$ is **coarser** than $\mathcal{T}'$.

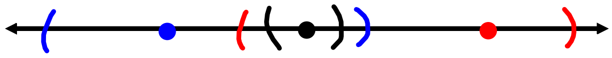
If X is a set, a **basis** for a topology on X is a collection $\mathfrak{B}$ of subsets of X such that:

1) for every $x \in X$, $\exists\, B \in \mathfrak{B}$ such that $x \in B$
2) Given $B_1, B_2 \in \mathfrak{B}$, if $x \in B_1 \cap B_2$, then $\exists\, B_3 \in \mathfrak{B}$ such that $x \in B_3 \subset B_1 \cap B_2$

A subset U of X is said to be **open** in X (that is, to be an element of $\mathcal{T}$) if for each $x \in U$, there is a basis element $B \in \mathfrak{B}$ such that $x \in B \subset U$.

$\mathfrak{B}$ is a basis for $\mathcal{T}$ if $\mathcal{T}$ is **generated** by $\mathfrak{B}$, i.e. for any $U \subseteq X$, U open, and $x \in U$, $\exists\, B \in \mathfrak{B}$ such that $x \in B \subset U$.



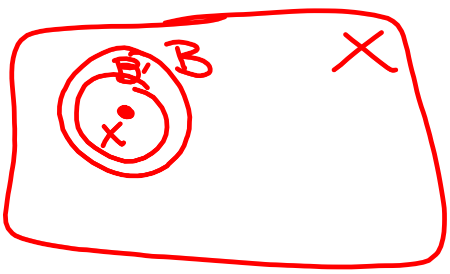


Lemma 13.1: Let X be a set; let $\mathfrak{B}$ be a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ equals the collection of all unions of elements of $\mathfrak{B}$.

Lemma 13.2: Let X be a topological space. Suppose that $\mathcal{C}$ is a collection of open sets of X such that for each open set U of X and each x in U, there is an element C of $\mathcal{C}$ such that $x \in C \subseteq U$. Then $\mathcal{C}$ is a basis for the topology of X.

Lemma 13.3: Let $\mathfrak{B}$ and $\mathfrak{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively on X. Then the following are equivalent:

1) $\mathcal{T}'$ is finer than $\mathcal{T}$ $\quad \mathcal{T} \subseteq \mathcal{T}'$
2) For each $x \in X$, and each basis element $B \in \mathfrak{B}$ containing x, there is a basis element $B' \in \mathfrak{B}'$ such that $x \in B' \subseteq B$

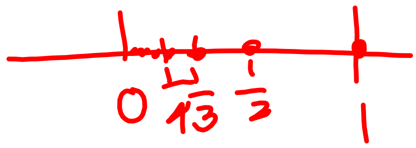


6. Show that the topologies of $\mathbb{R}_\ell$ and $\mathbb{R}_K$ are not comparable.

(show that neither is a subset of the other)

topology on $\mathbb{R}_\ell$ = "the lower limit topology"

$\tau_\ell \quad [a,b) = \{x \in \mathbb{R} \mid a \leq x < b\}$



topology on $\mathbb{R}_K$ = "the K-Topology"

$\tau_K \quad (a,b)$ together with $(a,b) - K$,
where $K = \{1/n \mid n \in \mathbb{Z}_+\} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots\}$

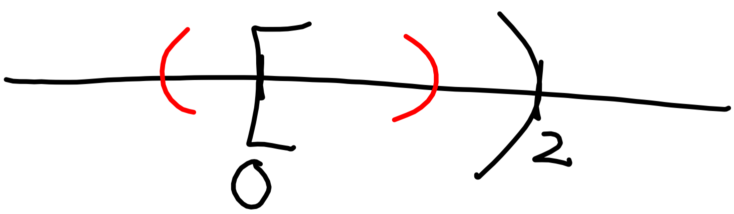
IS $\tau_\ell \subseteq \tau_K$ and/or $\tau_K \subseteq \tau_\ell$?

6. Show that the topologies of $\mathbb{R}_\ell$ and $\mathbb{R}_K$ are not comparable.
(show that neither is a subset of the other)
topology on $\mathbb{R}_\ell$ = "the lower limit topology"
$\tau_\ell \quad [a,b) = \{x \in \mathbb{R} \mid a \leq x < b\}$
topology on $\mathbb{R}_K$ = "the K-Topology"
$\tau_K \quad (a,b)$ together with $(a,b) - K$,
where $K = \{1/n \mid n \in \mathbb{Z}_+\}$

To show $\tau_1 \subseteq \tau_2$
Take $U_1 \in \tau_1$ & $x \in U_1$
and show $\exists\, U_2 \in \tau_2$
s.t. $x \in U_2 \subseteq U_1$

$\tau_\ell \not\subseteq \tau_K$ Take $[0,2) \in \tau_\ell$ and $0 \in [0,2)$

but there is no such (a,b) or $(a,b) - K$ in τ_K such that $0 \in (a,b)$ and $(a,b) \subseteq [0,2)$



6. Show that the topologies of $\mathbb{R}_\ell$ and $\mathbb{R}_K$ are not comparable. (show that neither is a subset of the other)

topology on $\mathbb{R}_\ell$ = "the lower limit topology"

τ_ℓ $[a,b) = \{x \in \mathbb{R} \mid a \le x < b\}$

topology on $\mathbb{R}_K$ = "the K-Topology"

τ_K (a,b) together with $(a,b)-K$, where $K = \{1/n \mid n \in \mathbb{Z}_+\}$

$$\tau_K \not\subseteq \tau_\ell$$

Take $(-1,1)-K \in \tau_K$ and $0 \in (-1,1)$.

but there is no such $[a,b) \in \tau_\ell$ such that $0 \in [a,b)$ and

$$[a,b) \subseteq (-1,1)-K$$

$[0,a)$

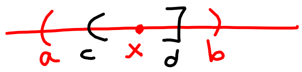
OR

$[a,b)$ $a<0<b$

τ_K (a,b) & $(a,b) - 1/n$

τ_4 = upper limit topology $(a,b]$

$\tau_K \subset \tau_4$



Case 1: Let $(a,b) \in \tau_K$ and $x \in (a,b)$.

take $c = \frac{x-a}{2}$ and $d = \frac{b-x}{2}$

Then $x \in (c,d] = \left(\frac{x-a}{2}, \frac{b-x}{2}\right] \subseteq (a,b)$.

Hence $\tau_K \subset \tau_4$.

Case 2: Let $(a,b)-K \in \tau_K$ and $x \in (a,b)-K$.

If $a>1$, $(a,b)-K = (a,b)$
If $b<0$, $(a,b)-K = (a,b)$ $\Big\}$ case 1

If $(a,b) \subseteq (0,1)$, take c to be the largest number of the form $\frac{1}{n}$ such that $x > c$.