A **topology** on a set X is a collection $\mathcal{T}$ of subsets of X with the following properties:

1) $\emptyset, X$ are in $\mathcal{T}$
2) the union of elements in *any* (arbitrary) subcollection of $\mathcal{T}$ is in $\mathcal{T}$

$$\bigcup_{\alpha} U_\alpha \in \mathcal{T}, \qquad U_\alpha \in \mathcal{T}\ \forall \alpha$$

3) the intersection of elements of any *finite* subcollection of $\mathcal{T}$ is in $\mathcal{T}$

$$\bigcap_{i=1}^{n} U_i \in \mathcal{T}, \qquad U_i \in \mathcal{T}\ \forall i$$

A set X with a specified $\mathcal{T}$ is called a **topological space**, denoted by $(X, \mathcal{T})$.

Let $(X, \mathcal{T})$ be a topological space, and $U \subseteq X$. U is said to be **open** if $U \in \mathcal{T}$.

discrete topology: the collection of all subsets of X (same as power set)

indiscrete or trivial topology: the collection consisting of X and $\emptyset$ only

finite complement topology: the collection $\mathcal{T}_f$ of all subsets U of X such that $X - U$ is either finite or all of X

co-countable topology: the collection $\mathcal{T}_C$ of all subsets U of X such that $X - U$ is either countable or all of X

Let X be a topological space and $Y \subset X$. Then the collection $\mathcal{T}_Y = \{U \cap X | U \text{ open in } X\}$ is a topology on Y, called the **subspace topology**.

standard topology on $\mathbb{R}$: topology generated by all open intervals $(a, b) = \{x \in \mathbb{R} | a < x < b\}$; whenever we consider $\mathbb{R}$, we assume it is given this topology unless we specifically state otherwise

the lower-limit topology on $\mathbb{R}$: topology generated by the collection $\mathbb{R}_\ell$ all half-open intervals of the form $[a, b) = \{x \in \mathbb{R} | a \leq x < b\}$

K-topology on $\mathbb{R}$: the topology generated by the collection $\mathbb{R}_K$ of all open intervals (a, b) along with all sets of the form $(a, b) - K$, where K is the set of all numbers of the form $1/n$, for $n \in \mathbb{Z}_+$

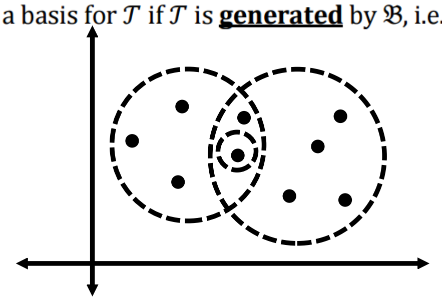
$\mathcal{T}$ is **comparable** with $\mathcal{T}'$ if $\mathcal{T}' \supseteq \mathcal{T}$ or $\mathcal{T} \supseteq \mathcal{T}'$.
If $\mathcal{T}' \supseteq \mathcal{T}$, we say that $\mathcal{T}'$ is **finer** than $\mathcal{T}$, and that $\mathcal{T}$ is **coarser** than $\mathcal{T}'$.

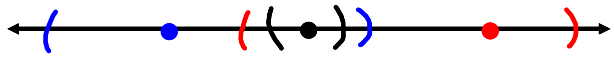
If X is a set, a **basis** for a topology on X is a collection $\mathfrak{B}$ of subsets of X such that:

1) for every $x \in X$, $\exists\, B \in \mathfrak{B}$ such that $x \in B$
2) Given $B_1, B_2 \in \mathfrak{B}$, if $x \in B_1 \cap B_2$, then $\exists\, B_3 \in \mathfrak{B}$ such that $x \in B_3 \subset B_1 \cap B_2$

A subset U of X is said to be **open** in X (that is, to be an element of $\mathcal{T}$) if for each $x \in U$, there is a basis element $B \in \mathfrak{B}$ such that $x \in B \subset U$.

$\mathfrak{B}$ is a basis for $\mathcal{T}$ if $\mathcal{T}$ is **generated** by $\mathfrak{B}$, i.e. for any $U \subseteq X$, U open, and $x \in U$, $\exists\, B \in \mathfrak{B}$ such that $x \in B \subset U$.



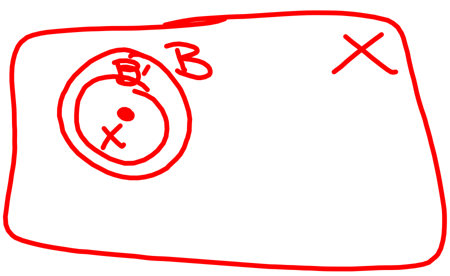


Lemma 13.1: Let X be a set; let $\mathfrak{B}$ be a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ equals the collection of all unions of elements of $\mathfrak{B}$.

Lemma 13.2: Let X be a topological space. Suppose that $\mathcal{C}$ is a collection of open sets of X such that for each open set U of X and each x in U, there is an element C of $\mathcal{C}$ such that $x \in C \subseteq U$. Then $\mathcal{C}$ is a basis for the topology of X.

Lemma 13.3: Let $\mathfrak{B}$ and $\mathfrak{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively on X. Then the following are equivalent:

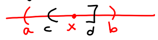
1) $\mathcal{T}'$ is finer than $\mathcal{T}$ $\quad \mathcal{T} \subseteq \mathcal{T}'$
2) For each $x \in X$, and each basis element $B \in \mathfrak{B}$ containing x, there is a basis element $B' \in \mathfrak{B}'$ such that $x \in B' \subseteq B$



$\mathcal{T}_K$ (a,b) & $(a,b) - 1/n$

$\mathcal{T}_4$ = upper limit topology $(a,b]$

$\mathcal{T}_K \subset \mathcal{T}_4$



Case 1: Let $(a,b) \in \mathcal{T}_K$ and $x \in (a,b)$.

take $c = \frac{x-a}{2}$ and $d = \frac{b-x}{2}$

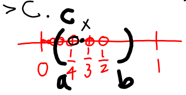
Then $x \in (c,d] = \left(\frac{x-a}{2}, \frac{b-x}{2}\right] \subseteq (a,b)$.

Hence $\mathcal{T}_K \subset \mathcal{T}_4$.

Case 2: Let $(a,b)-K \in \mathcal{T}_K$ and $x \in (a,b)-K$.

$\left.\begin{array}{l}\text{If } a > 1, \ (a,b)-K = (a,b) \\ \text{If } b < 0, \ (a,b)-K = (a,b)\end{array}\right\}$ case 1

If $(a,b) \subseteq (0,1)$, take c to be the largest number of the form $\frac{1}{n}$ Such that $x > c$.



Take

so that

$x \in \underbrace{(c, x]}_{} \subseteq (a,b)$

$\mathcal{T}_4 \overset{?}{\subseteq} \mathcal{T}_K$

take $(-1, 0]$ and $0 \in (-1, 0]$

There are no (a,b) or $(a,b) - \frac{1}{n}$ in $\mathcal{T}_K$ s.t. $0 \in (a,b) \subseteq (-1, 0]$

$\mathcal{T}_K \subsetneq \mathcal{T}_4$

$\mathcal{T}_\ell$ & $\mathcal{T}_K$ are not comparable

3. Show that the collection $\mathcal{T}_c$ given in Example 4 of §12 is a topology on the set X. Is the collection

$$\mathcal{T}_\infty = \{U \mid X - U \text{ is infinite or empty or all of } X\}$$

a topology on X? ← No. Why?

$\mathcal{T}_c$ = cocountable topology

$= \{U \mid X - U \text{ is either countable or all of } X\}$

To show:

1) $\emptyset, X \in \mathcal{T}_c$

2) $\bigcup_\alpha U_\alpha \in \mathcal{T}_c$

3) $\bigcap_{i=1}^{n} U_i \in \mathcal{T}_c$

$\left.\begin{array}{l} X - \bigcup_\alpha U_\alpha \\ X - \bigcap_{i=1}^{n} U_i \end{array}\right\}$ use DeMorgan's Laws

7. Consider the following topologies on $\mathbb{R}$:

$\mathcal{T}_1$ = the standard topology, (a,b)

$\mathcal{T}_2$ = the topology of $\mathbb{R}_K$, $(a,b) + (a,b)-K$

$\mathcal{T}_3$ = the finite complement topology, $\{U \mid X-U \text{ is finite or all of } X\}$

$\mathcal{T}_4$ = the upper limit topology, having all sets $(a,b]$ as basis,

$\mathcal{T}_5$ = the topology having all sets $(-\infty, a) = \{x \mid x < a\}$ as basis.

Determine, for each of these topologies, which of the others it contains.

$\mathcal{T}_2 \subsetneq \mathcal{T}_4$: the topology on $\mathbb{R}_K$ is strictly coarser

Closed Sets, Limit Points, and Continuity

Recall: U is open in X if U is an element of the topology $\mathcal{T}_X$ on X

Def Let X be a topological space. $A \subset X$ is a **closed set** if $X - A$ ia open.

Examples:

$\emptyset$: $X - \emptyset = X$ open $\Rightarrow \emptyset$ is closed

X : $X - X = \emptyset$ open $\Rightarrow X$ is closed

$\emptyset, X$ are both open and closed

$[a,b]$ in $\mathbb{R}$ w/ standard topology

$\mathbb{R}-[a,b] = (-\infty,a) \cup (b,\infty)$

union of open intervals is open

$[a,b]$ is closed

why is (b,∞) open?

$$\bigcup_{n=1}^{\infty} \underbrace{(b,b+n)}_{\text{open in } \mathbb{R}} = (b,\infty)$$

arbitrary union of open sets

$(-\infty,a]$?

$\mathbb{R}-(-\infty,a] = (a,\infty)$ open

$\Rightarrow (-\infty,a]$ closed

co-finite topology on X

$\{U \mid X-U$ is either finite or all of $X\}$

$A=\{a\}\subseteq X$ one element set

$X-A=X-\{a\}$

$X-(X-\{a\})=\{a\}$ finite

$\Rightarrow X-\{a\}$ is open

$\Rightarrow \{a\}$ is closed

$\tau_1 \subseteq \tau_2$

If given $B_1\in\tau_1$ and $x\in B_1$

we can find $B_2\in\tau$ such

that $x\in B_2\subseteq B_1$

Thm 17.1 Let X be a topological space. The following hold:

1) $\emptyset, X$ are closed
2) arbitrary intersections of closed sets are closed, i.e. if A_i are closed, $\bigcap_i A_i$ is closed.
3) finite unions of closed sets are closed

2) $X - \bigcap_\alpha U_\alpha = \bigcup_\alpha (X - U_\alpha)$

If all U_α's are closed, $X - U_\alpha$'s are open & arbitrary union of open sets is open

3. $X - \bigcup_{i=1}^n U_i = \bigcap_{i=1}^n (X - U_i)$