A **topology** on a set X is a collection $\mathcal{T}$ of subsets of X with the following properties:

1) $\emptyset, X$ are in $\mathcal{T}$
2) the union of elements in *any* (arbitrary) subcollection of $\mathcal{T}$ is in $\mathcal{T}$
$$\bigcup_{\alpha} U_\alpha \in \mathcal{T}, \qquad U_\alpha \in \mathcal{T}\ \forall \alpha$$
3) the intersection of elements of any *finite* subcollection of $\mathcal{T}$ is in $\mathcal{T}$
$$\bigcap_{i=1}^{n} U_i \in \mathcal{T}, \qquad U_i \in \mathcal{T}\ \forall i$$

A set X with a specified $\mathcal{T}$ is called a **topological space**, denoted by $(X, \mathcal{T})$.

Let $(X, \mathcal{T})$ be a topological space, and $U \subseteq X$. U is said to be **open** if $U \in \mathcal{T}$.

$X = \{a, b, c\}$

$\mathcal{T} = \{\emptyset, X, \{a,b\}, \{b,c\}\}$

not a topology on X

$\{a,b\} \cap \{b,c\} = \{b\} \notin \mathcal{T}$

$\mathcal{T}_+ = \{\emptyset, X, \{a,b\}, \{b,c\}, \{b\}\}$

is a topology on X

$\mathcal{T}_{++} = \{\emptyset, X, \{b\}\}$

Trivial topology on X

$= \{\emptyset, X\}$

discrete topology: the collection of all subsets of X (same as power set)

indiscrete or trivial topology: the collection consisting of X and $\emptyset$ only

finite complement topology: the collection $\mathcal{T}_f$ of all subsets U of X such that $X - U$ is either finite or all of X

co-countable topology: the collection $\mathcal{T}_C$ of all subsets U of X such that $X - U$ is either countable or all of X

Let X be a topological space and $Y \subset X$. Then the collection $\mathcal{T}_Y = \{U \cap X | U \text{ open in } X\}$ is a topology on Y, called the **subspace topology**.

standard topology on $\mathbb{R}$: topology generated by all open intervals $(a, b) = \{x \in \mathbb{R} | a < x < b\}$; whenever we consider $\mathbb{R}$, we assume it is given this topology unless we specifically state otherwise

the lower-limit topology on $\mathbb{R}$: topology generated by the collection $\mathbb{R}_\ell$ all half-open intervals of the form $[a, b) = \{x \in \mathbb{R} | a \le x < b\}$

K-topology on $\mathbb{R}$: the topology generated by the collection $\mathbb{R}_K$ of all open intervals (a, b) along with all sets of the form $(a, b) - K$, where K is the set of all numbers of the form $1/n$, for $n \in \mathbb{Z}_+$

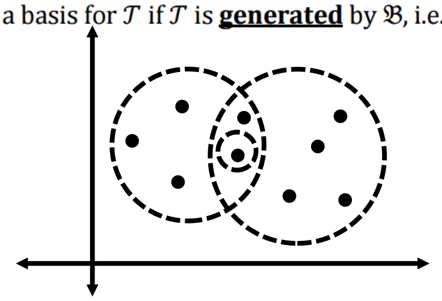
$\mathcal{T}$ is **comparable** with $\mathcal{T}'$ if $\mathcal{T}' \supseteq \mathcal{T}$ or $\mathcal{T} \supseteq \mathcal{T}'$.
If $\mathcal{T}' \supseteq \mathcal{T}$, we say that $\mathcal{T}'$ is **finer** than $\mathcal{T}$, and that $\mathcal{T}$ is **coarser** than $\mathcal{T}'$.

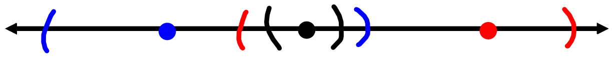
If X is a set, a **basis** for a topology on X is a collection $\mathfrak{B}$ of subsets of X such that:

1) for every $x \in X$, $\exists\, B \in \mathfrak{B}$ such that $x \in B$
2) Given $B_1, B_2 \in \mathfrak{B}$, if $x \in B_1 \cap B_2$, then $\exists\, B_3 \in \mathfrak{B}$ such that $x \in B_3 \subset B_1 \cap B_2$

A subset U of X is said to be **open** in X (that is, to be an element of $\mathcal{T}$) if for each $x \in U$, there is a basis element $B \in \mathfrak{B}$ such that $x \in B \subset U$.

$\mathfrak{B}$ is a basis for $\mathcal{T}$ if $\mathcal{T}$ is **generated** by $\mathfrak{B}$, i.e. for any $U \subseteq X$, U open, and $x \in U$, $\exists\, B \in \mathfrak{B}$ such that $x \in B \subset U$.





Lemma 13.1: Let X be a set; let $\mathfrak{B}$ be a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ equals the collection of all unions of elements of $\mathfrak{B}$.

Lemma 13.2: Let X be a topological space. Suppose that $\mathcal{C}$ is a collection of open sets of X such that for each open set U of X and each x in U, there is an element C of $\mathcal{C}$ such that $x \in C \subseteq U$. Then $\mathcal{C}$ is a basis for the topology of X.

Lemma 13.3: Let $\mathfrak{B}$ and $\mathfrak{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively on X. Then the following are equivalent:

1) $\mathcal{T}'$ is finer than $\mathcal{T}$ $\quad \tau \subseteq \tau'$
2) For each $x \in X$, and each basis element $B \in \mathfrak{B}$ containing x, there is a basis element $B' \in \mathfrak{B}'$ such that $x \in B' \subseteq B$

Closed Sets, Limit Points, and Continuity

Recall: U is open in X if U is an element of the topology $\mathcal{T}_X$ on X

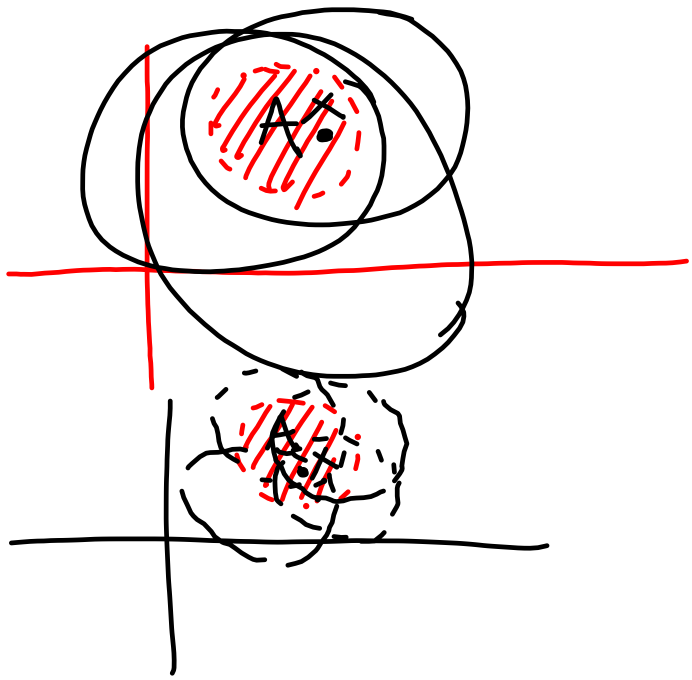
Def Let X be a topological space. $A \subset X$ is a **closed set** if $X - A$ ia open.

Thm 17.1 Let X be a topological space. The following hold:

1) $\emptyset$, X are closed
2) arbitrary intersections of closed sets are closed, i.e. if A_i are closed, $\bigcap_i A_i$ is closed.
3) finite unions of closed sets are closed

Def Let X be a topological space and let $A \subset X$. The **dosure** of A, denoted by $\bar{A}$, is the intersection of all closed sets containing A.

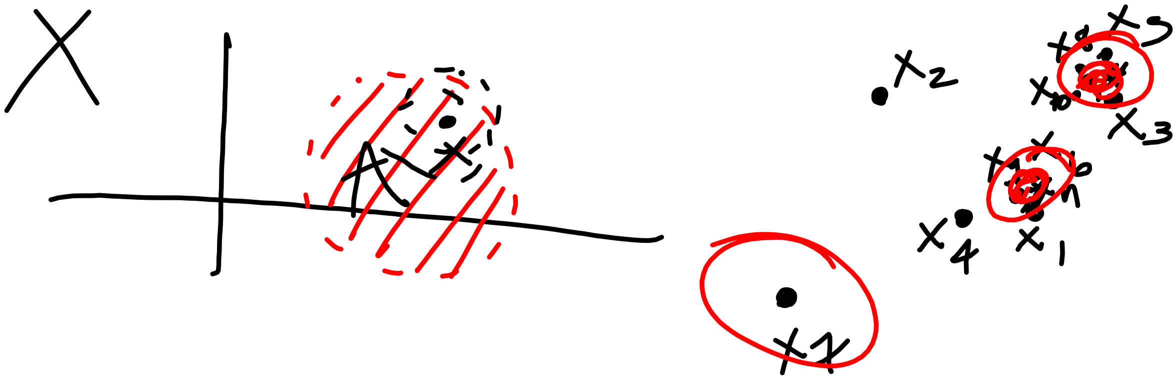
Thm 17.5 Let $A \subset X$ and let $\mathfrak{B}$ be a basis for X. Then $x \in \bar{A}$ if and only if every open set U containing x intersects A and $x \in \bar{A}$ if and only if every basis element B containing x intersects A



$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$
= cartesian plane

Def U is a **neighborhood** of x if U is open and $x \in U$

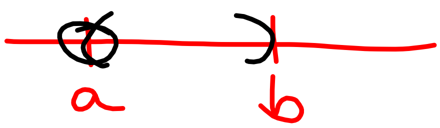
Def Let X be a topological space and let $A \subset X$. $x \in X$ is said to be a **limit point** of A if every neighborhood of x intersects A in a point other than x.



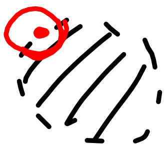
Thm 17.6 Let $A \subset X$ and let A' be the set of all limit points of A. Then $\bar{A} = A \cup A'$.

Cor 17.7 A is closed if and only if A contains all its limit points.

(a,b) limit points are a & b

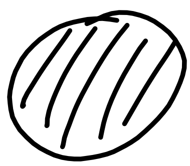


$\overline{(a,b)} = [a,b]$



U = open disk

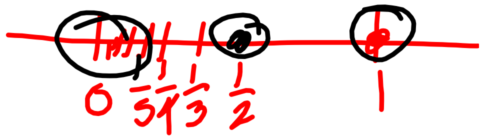
$\overline{U}$ = closed disk



$A = (0,1]$

$\bar{A} = [0,1]$

$B = \{\frac{1}{n} \mid n \in \mathbb{Z}_+\}$



$\bar{B} = B \cup \{0\}$

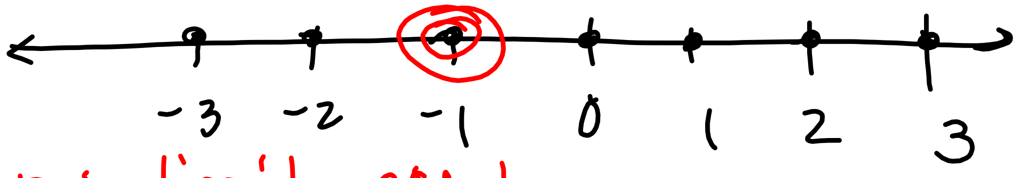
$\mathbb{Q} = \{p/q \mid p, q \in \mathbb{Z}\}$

= rational #'s

$\bar{\mathbb{Q}} = \mathbb{R}$

$\mathbb{Z}$ = integers

$\overline{\mathbb{Z}} = \mathbb{Z}$



no limit points

Thm 17.6 Let $A \subset X$ and let A' be the set of all limit points of A. Then $\bar{A} = A \cup A'$.

Cor 17.7 A is closed if and only if A contains all its limit points.

Proof:

$\Rightarrow$ suppose A is closed $\Rightarrow A = \bar{A}$

$A \subseteq A \cup A' = \bar{A} \Rightarrow A' \subseteq A$

$\Rightarrow$ A contains all of its limit points

$\Leftarrow$ suppose A contains all its limit points.

$A' \subseteq A$, $A \subseteq A$; $A' \cup A \subseteq A$

$A' \cup A = \bar{A} \subseteq A$ $\quad A \subseteq \bar{A} \Rightarrow A = \bar{A}$

$\Rightarrow A$ is closed