A **topology** on a set X is a collection $\mathcal{T}$ of subsets of X with the following properties:

1) $\emptyset, X$ are in $\mathcal{T}$
2) the union of elements in *any* (arbitrary) subcollection of $\mathcal{T}$ is in $\mathcal{T}$

$$\bigcup_{\alpha} U_\alpha \in \mathcal{T}, \qquad U_\alpha \in \mathcal{T}\ \forall \alpha$$

3) the intersection of elements of any *finite* subcollection of $\mathcal{T}$ is in $\mathcal{T}$

$$\bigcap_{i=1}^{n} U_i \in \mathcal{T}, \qquad U_i \in \mathcal{T}\ \forall i$$

A set X with a specified $\mathcal{T}$ is called a **topological space**, denoted by $(X, \mathcal{T})$.

Let $(X, \mathcal{T})$ be a topological space, and $U \subseteq X$. U is said to be **open** if $U \in \mathcal{T}$.

discrete topology: the collection of all subsets of X (same as power set)

indiscrete or trivial topology: the collection consisting of X and $\emptyset$ only

finite complement topology: the collection $\mathcal{T}_f$ of all subsets U of X such that $X - U$ is either finite or all of X

co-countable topology: the collection $\mathcal{T}_C$ of all subsets U of X such that $X - U$ is either countable or all of X

Let X be a topological space and $Y \subset X$. Then the collection $\mathcal{T}_Y = \{U \cap X | U \textit{ open in } X\}$ is a topology on Y, called the **subspace topology**.

standard topology on $\mathbb{R}$: topology generated by all open intervals $(a, b) = \{x \in \mathbb{R} | a < x < b\}$; whenever we consider $\mathbb{R}$, we assume it is given this topology unless we specifically state otherwise

the lower-limit topology on $\mathbb{R}$: topology generated by the collection $\mathbb{R}_\ell$ all half-open intervals of the form $[a, b) = \{x \in \mathbb{R} | a \leq x < b\}$

K-topology on $\mathbb{R}$: the topology generated by the collection $\mathbb{R}_K$ of all open intervals (a, b) along with all sets of the form $(a, b) - K$, where K is the set of all numbers of the form $1/n$, for $n \in \mathbb{Z}_+$

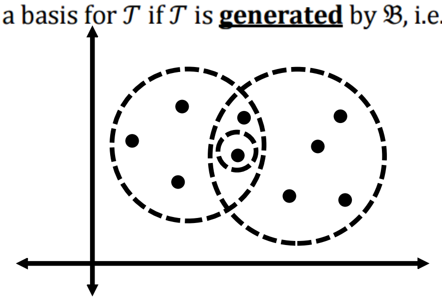
$\mathcal{T}$ is **comparable** with $\mathcal{T}'$ if $\mathcal{T}' \supseteq \mathcal{T}$ or $\mathcal{T} \supseteq \mathcal{T}'$.
If $\mathcal{T}' \supseteq \mathcal{T}$, we say that $\mathcal{T}'$ is **finer** than $\mathcal{T}$, and that $\mathcal{T}$ is **coarser** than $\mathcal{T}'$.

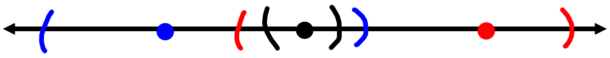
If X is a set, a **basis** for a topology on X is a collection $\mathfrak{B}$ of subsets of X such that:

1) for every $x \in X$, $\exists\, B \in \mathfrak{B}$ such that $x \in B$
2) Given $B_1, B_2 \in \mathfrak{B}$, if $x \in B_1 \cap B_2$, then $\exists\, B_3 \in \mathfrak{B}$ such that $x \in B_3 \subset B_1 \cap B_2$

A subset U of X is said to be **open** in X (that is, to be an element of $\mathcal{T}$) if for each $x \in U$, there is a basis element $B \in \mathfrak{B}$ such that $x \in B \subset U$.

$\mathfrak{B}$ is a basis for $\mathcal{T}$ if $\mathcal{T}$ is **generated** by $\mathfrak{B}$, i.e. for any $U \subseteq X$, U open, and $x \in U$, $\exists\, B \in \mathfrak{B}$ such that $x \in B \subset U$.





Lemma 13.1: Let X be a set; let $\mathfrak{B}$ be a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ equals the collection of all unions of elements of $\mathfrak{B}$.

Lemma 13.2: Let X be a topological space. Suppose that $\mathcal{C}$ is a collection of open sets of X such that for each open set U of X and each x in U, there is an element C of $\mathcal{C}$ such that $x \in C \subseteq U$. Then $\mathcal{C}$ is a basis for the topology of X.

Lemma 13.3: Let $\mathfrak{B}$ and $\mathfrak{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively on X. Then the following are equivalent:

1) $\mathcal{T}'$ is finer than $\mathcal{T}$ $\quad \mathcal{T} \subseteq \mathcal{T}'$
2) For each $x \in X$, and each basis element $B \in \mathfrak{B}$ containing x, there is a basis element $B' \in \mathfrak{B}'$ such that $x \in B' \subseteq B$

Closed Sets, Limit Points, and Continuity

Recall: U is open in X if U is an element of the topology $\mathcal{T}_X$ on X

Def Let X be a topological space. $\mathrm{A} \subset X$ is a **closed set** if $X - A$ ia open.

Thm 17.1 Let X be a topological space. The following hold:

1) $\emptyset, X$ are closed
2) arbitrary intersections of closed sets are closed, i.e. if A_i are closed, $\bigcap_i A_i$ is closed.
3) finite unions of closed sets are closed

Def Let X be a topological space and let $\mathrm{A} \subset X$. The **dosure** of A, denoted by $\bar{A}$, is the intersection of all closed sets containing A.

Thm 17.5 Let $\mathrm{A} \subset X$and let $\mathfrak{B}$ be a basis for X. Then $x \in \bar{A}$ if and only if every open set U containing x intersects A and $x \in \bar{A}$ if and only if every basis element B containing x intersects A

Def U is a **neighborhood** of x if U is open and $x \in U$

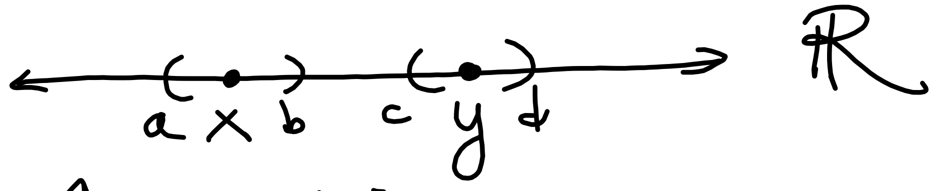
Def Let X be a topological space and let $\mathrm{A} \subset X$. $x \in X$ is said to be a **limit point** of A if every neighborhood of x intersects A in a point other than x.

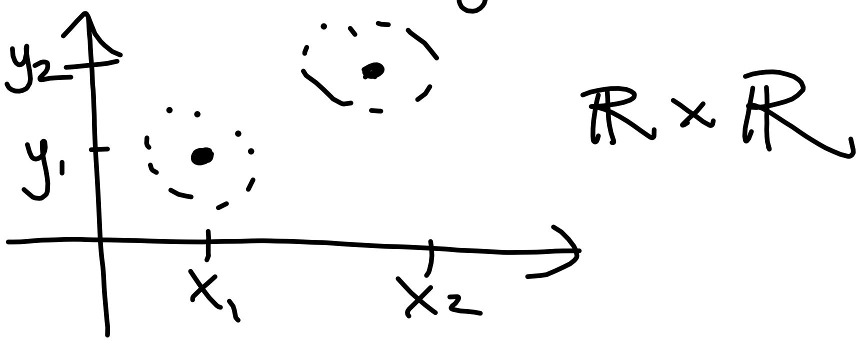
Thm 17.6 Let $\mathrm{A} \subset X$ and let A' be the set of all limit points of A. Then $\bar{A} = A \cup A'$.

Cor 17.7 A is closed if and only if A contains all its limit points.

A is closed iff $A = \bar{A}$

Def A topological space is **Hausdorff** if for each pair of distinct points x and y, there exist disjoint neighborhoods of x and y.





X with the trivial topology

$\tau = \{\emptyset, X\}$

Given any $x \neq y \in X$

Clearly not Hausdoff, as X is the only open set containing either x or y

cofinite topology on $\mathbb{Z}_+$

$\{U \mid \mathbb{Z}_+ - U \text{ is finite or all of } \mathbb{Z}_+\}$

$x = 1,\ y = 2 \in \mathbb{Z}_+$

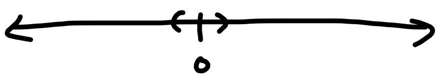
not Hausdorff

Thm 17.8 Every finite set in a Hausdorff space is closed.* finite union of single point sets

Thm 17.9 Let X be a Hausdorff space and let $A \subset X$. Then x is a limit point of A if and only if every neighborhood of x contains infinitely many points of A.

*finite sets being closed is weaker condition than Hausdorff, and is called the **T_1-axiom***

e.g. the set of positive integers with the co-finite topology satisfies the T_1-axiom but is not Hausdorff



X-Hausdorff space ; $A = \{a\}$, $A \subseteq X$

Def A topological space is **Hausdorff** if for each pair of distinct points x and y, there exist disjoint neighborhoods of x and y.

finite union of one-element sets

Thm 17.8 Every finite set in a Hausdorff space is closed.

A subset U of X is said to be **open** in X (that is, to be an element of $\mathcal{T}$) if for each $x \in U$, there is a basis element $B \in \mathfrak{B}$ such that $x \in B \subset U$.

1. Let X be a topological space; let A be a subset of X. Suppose that for each $x \in A$ there is an open set U containing x such that $U \subset A$. Show that A is open in X.

Let $A = \{a\}$. $X - A = X - \{a\}$.

Let $b \in X$.

Since X is Hausdoff, there exist disjoint neighborhoods (open sets) U & V such that $a \in U$, $b \in V$ and $U \cap V = \emptyset$

$V \subseteq X - \{a\} \Rightarrow b \in X - \{a\}$

V is an open set in X such that $b \in V \subseteq X - \{a\} \Rightarrow X - \{a\}$ is open in X

$\Rightarrow \{a\}$ is closed.

Def Let X and Y be topological spaces and $f: X \to Y$ be a function. f is said to be **continuous** if for every open set V in Y, $f^{-1}(V)$ is open in X.
$f^{-1}(V) = \{x \in X | f(x) \in V\}$ (pre-images of open sets are open)

If $\mathcal{B}$ is a basis for Y, then f is continuous if $f^{-1}(B)$ is open for all $B \in \mathcal{B}$.

Examples:

1. $f: X \to Y$, $f(x) = y_0 \in Y$ for all $x \in X$ (constant function)

Let V be an open set in Y.

$$f^{-1}(V) = \begin{cases} X & \text{if } y_0 \in V \\ \emptyset & \text{if } y_0 \notin V \end{cases}$$

both X & $\emptyset$ are open in X

Hence f is continuous

2. $\pi_1 : X \times Y \longrightarrow X$ $\qquad$ $\pi_2 : X \times Y \to Y$

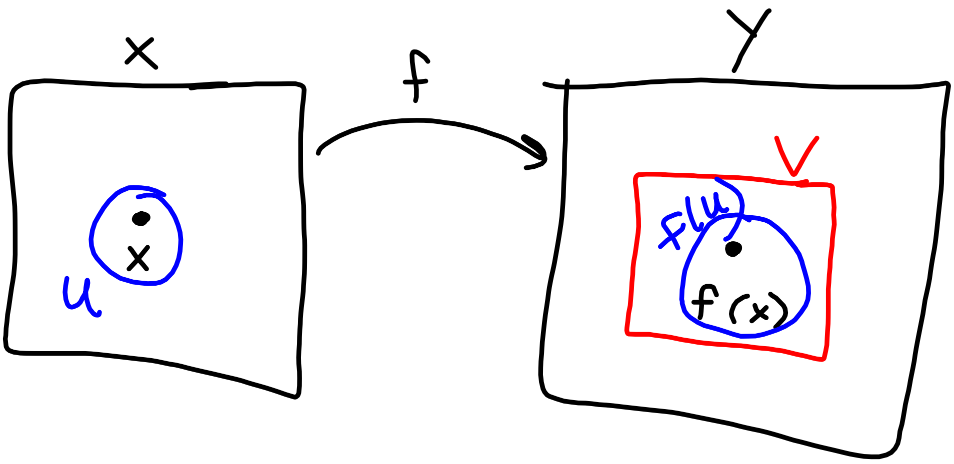
$\pi_1(x, y) = x$ projection onto first coordinate

Let V be an open set in X

$\pi_1^{-1}(V) = V \times Y$ cross-product of open sets is open

Thm 18.1 Let X and Y be topological spaces and $f : X \to Y$ be a function. The following are equivalent:

1) f is continuous
2) $f^{-1}(B)$ is closed for every closed set $B \subset Y$
3) for each $x \in X$ and each neighborhood V of $f(x)$, there exists a neighborhood U of x such that $f(U) \subset V$.



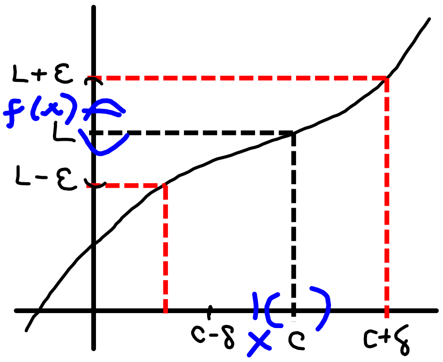
$\varepsilon - \delta$ Definition of the Limit:

Let f be a function defined on an open interval containing c (except possibly at c) and let L be a real number. The statement

$$\lim_{x \to c} f(x) = L$$

means that for each $\varepsilon > 0$, there exists a $\delta > 0$ such that if $0 < |x - c| < \delta$, then $|f(x) - L| < \varepsilon$.

f is continuous if
$f(c) = L$



Def Let $f: X \to Y$ be a bijection. If both f and f^{-1} are continuous, then f is called a **homeomorphism**.