1. Let X be a topological space; let A be a subset of X. Suppose that for each $x \in A$ there is an open set U containing x such that $U \subset A$. Show that A is open in X.

A subset U of X is said to be **open** in X (that is, to be an element of $\mathcal{T}$) if for each $x \in U$, there is a basis element $B \in \mathfrak{B}$ such that $x \in B \subset U$.

Given: Topological space X;
$A \subseteq X$
$\forall x \in A, \exists\ U \in \mathcal{T}_X$ such that $U \subseteq A$.
To show: $A \in \mathcal{T}_X$. (A is open in X).

Let $\mathcal{B}$ be a basis for $\mathcal{T}_X$.
U open in $X \Rightarrow \forall x \in U, \exists\ B \in \mathcal{B}$ such that $x \in B \subseteq U$.
For A to be open in X, for all $x \in A$ we need a $B \in \mathcal{B}$ such that $x \in B \subseteq A$.
Since we have that $\forall x \in A, \exists\ U \in \mathcal{T}_X$ such that $x \in U \subseteq A$

Proof:
Let $x \in A$. $\exists\ U \in \mathcal{T}_X$ such that $x \in U \subseteq A$.
$\Rightarrow \exists\ B \in \mathcal{B}$ such that $x \in B \subseteq U \subseteq A$
$\Rightarrow A \in \mathcal{T}_X$.

3. Show that the collection $\mathcal{T}_c$ given in Example 4 of §12 is a topology on the set X.

co-countable topology: the collection $\mathcal{T}_C$ of all subsets U of X such that $X - U$ is either countable or all of X

A **topology** on a set X is a collection $\mathcal{T}$ of subsets of X with the following properties:

1) $\emptyset, X$ are in $\mathcal{T}$
2) the union of elements in *any* (arbitrary) subcollection of $\mathcal{T}$ is in $\mathcal{T}$

$$\bigcup_\alpha U_\alpha \in \mathcal{T}, \qquad U_\alpha \in \mathcal{T}\ \forall \alpha$$

3) the intersection of elements of any *finite* subcollection of $\mathcal{T}$ is in $\mathcal{T}$

$$\bigcap_{i=1}^n U_i \in \mathcal{T}, \qquad U_i \in \mathcal{T}\ \forall i$$

1) $\emptyset$: $X - \emptyset = X$, which is all of $X \Rightarrow \emptyset \in \mathcal{T}_c$
X: $X - X = \emptyset$, which is countable $\Rightarrow X \in \mathcal{T}_c$

2) $\bigcup_\alpha U_\alpha$: $X - \bigcup_\alpha U_\alpha = \bigcap_\alpha (X - U_\alpha)$
Since $U_\alpha \in \mathcal{T}_c\ \forall \alpha$, $X - U_\alpha$ is either countable or all of X.
Case 1: $X - U_\alpha$ is all of $X\ \forall \alpha$ $(U_\alpha = \emptyset\ \forall \alpha)$
(Case 2: $X - U_\alpha = \emptyset$ for at least one α $(U_\alpha = X$ for at least one $\alpha)$) — $\emptyset$ countable
Case 3: $U_\alpha \neq \emptyset$ for at least one α (and $U_\alpha \neq X\ \forall \alpha$)
$\Rightarrow X - U_\alpha$ is countable
$\bigcap_\alpha (X - U_\alpha) \subseteq X - U_\alpha\ \forall \alpha$
subsets of countable sets are countable
$\Rightarrow \bigcup_\alpha U_\alpha \in \mathcal{T}_c$

3) $\bigcap_{i=1}^n U_i$: $X - \bigcap_{i=1}^n U_i = \bigcup_{i=1}^n (X - U_i)$
Case 1: $U_i = \emptyset$ for at least one i.
$X - U_i = X \Rightarrow \bigcup_{i=1}^n (X - U_i) = X$
Case 2: $U_i \neq \emptyset\ \forall i$
$\Rightarrow X - U_i$ is countable
$\bigcup_{i=1}^n (X - U_i)$ is a finite union of countable sets and hence countable
$\Rightarrow \bigcap_{i=1}^n U_i \in \mathcal{T}_c$

Is the collection

$$\mathcal{T}_\infty = \{U \mid X - U \text{ is infinite or empty or all of } X\}$$

a topology on X?

7. Consider the following topologies on $\mathbb{R}$:

$\mathcal{T}_1$ = the standard topology, $(a, b) = \{x \in \mathbb{R} | a < x < b\}$;
$\mathcal{T}_2$ = the topology of $\mathbb{R}_K$, $(a, b) - 1/n$, for $n \in \mathbb{Z}_+$
$\mathcal{T}_3$ = the finite complement topology, $\mathcal{T}_f$ $X - U$ is either finite or all of X
$\mathcal{T}_4$ = the upper limit topology, having all sets $(a, b]$ as basis,
$\mathcal{T}_5$ = the topology having all sets $(-\infty, a) = \{x \mid x < a\}$ as basis.

Determine, for each of these topologies, which of the others it contains.

We showed that the lower limit topology is not comparable with the k-topology. (#6)

We have also already shown that $\mathcal{T}_4$ is strictly finer than $\mathcal{T}_2$ (see class notes from 4/25)

$$\mathcal{T}_4 \not\supsetneq \mathcal{T}_2$$

8. (a) Apply Lemma 13.2 to show that the countable collection

$$\mathcal{B} = \{(a, b) \mid a < b, a \text{ and } b \text{ rational}\}$$

is a basis that generates the standard topology on $\mathbb{R}$.

standard topology on $\mathbb{R}$: topology generated by all open intervals $(a, b) = \{x \in \mathbb{R} | a < x < b\}$; whenever we consider $\mathbb{R}$, we assume it is given this topology unless we specifically state otherwise

Lemma 13.2: Let X be a topological space. Suppose that $\mathcal{C}$ is a collection of open sets of X such that for each open set U of X and each x in U, there is an element C of $\mathcal{C}$ such that $x \in C \subseteq U$. Then $\mathcal{C}$ is a basis for the topology of X.

(b) Show that the collection

$$\mathcal{C} = \{[a, b) \mid a < b, a \text{ and } b \text{ rational}\}$$

is a basis that generates a topology different from the lower limit topology on $\mathbb{R}$.

the lower-limit topology on $\mathbb{R}$: topology generated by the collection $\mathbb{R}_\ell$ all half-open intervals of the form $[a, b) = \{x \in \mathbb{R} | a \leq x < b\}$