The Product Topology

also known as Tychonoff Topology

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6th Period Topology

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Topology Vocab:

Cartesian Product: The Cartesian product of two sets *A* and *B* (also called the product set, set direct product, or cross product) is defined to be the set of all points *(a,b)* where *a is an element of A* and *b is an element of B*.

Topological Space: A topological space, also called an abstract topological space, is a set *X* together with a collection of open subsets *T* that satisfies the four conditions:

1. The empty set is in *T*.
2. *X* is in *T*.
3. The intersection of a finite number of sets in *T* is also in *T*.
4. The union of an arbitrary number of sets in *T* is also in *T*.

Real Line: Most commonly, "real line" is used to mean real axis, i.e., a line with a fixed scale so that every real number corresponds to a unique point on the line.

Euclidean Topology: A metric topology induced by the Euclidean metric. In the Euclidean topology of the *n*-dimensional space *all Real Numbers raised to the n*, the open sets are the unions of *n* balls.

Euclidean Space: Euclidean *n*-space, sometimes called Cartesian space or simply *n*-space, is the space of all *n*-tuples of real numbers, (*x1*, *x2*, ..., *xn*)

The definition:

The topology on the Cartesian product *X* x *Y* of two topological spaces whose open sets are the unions of subsets *A* x *B*, where *A* and *B* are open subsets of *X* and *Y*, respectively.

This definition extends in a natural way to the Cartesian product of any finite number *n* of topological spaces. The product topology of

R x... x R,
n times

where R the real line with the Euclidean topology, coincides with the Euclidean topology of the Euclidean space *all real numbers raised to the n.*

Munkres 2.15 The Product Topology on *X* x *Y*

Definition: Let *X* and *Y* be topological spaces. The ***product*** topology on *X* x *Y* is the topology having as basis the collection of

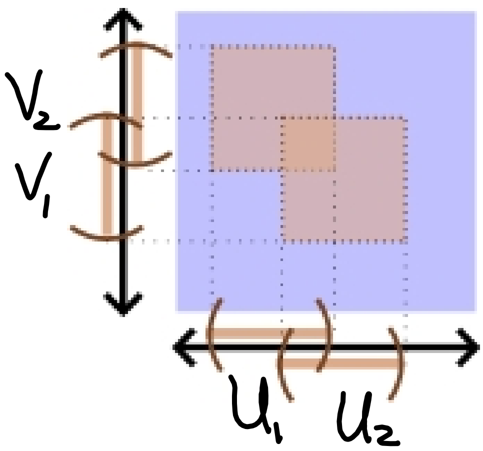
$\mathcal{B}$ *B (script B)* of all sets of the form *U* x *V,* where *U* is an open subset of *X* and *V* is an open subset of *Y.*

Next check that *B (script B)* is a basis. You can do this by acknowledging that:

1. The first condition is trivial, since *X* x *Y* is itself a basis element.

2. The second condition says that the intersection of two basis elements is a basis element because *U1* x *V1* and *U2* x *V2* is another basis element.

3. For the intersection of (*U1* x *V1) and* (*U2* x *V2)*= (*U1 and U2) x* (*V1 and V2),* the latter set is a basis element because *U1* and *U2* are open in *X* and *Y,* respectively.



It is important to note that the collection *B (script B)* is not a topology on *X* x *Y.* The union of the two rectangles above is not a product of two sets, so it cannot belong to *B (script B); however it is open in X* x *Y* . So if two topologies are given by bases you can say:

Theorem 15.1 *If B (script B) is a basis for the topology of X and C (script C) is a basis for the topology of Y, then the collection*

D (script D)= {B x *C| B is an element of script B* and *C is an element of script C}*

is a basis for the topology of X x *Y.*

How do we prove all of this??

We use Lemma 13.2 which states

Let X be a topological space. Suppose that C (script C) is a collection of open sets of X such that for each open set U of X and each x in U, there is an element C of script C such that x is an element of C which is a subset of U. Then script C is a basis for the topology of X.

And what does all of that actually mean?

It means we need to show that two conditions ~~from~~ form the definition of basis hold for script B.

Condition 1: Since A1 is an element of *T1* and A2 is an element of *T2,* A1 x A2 is an element of P.

Condition 2: Suppose U1, V1 is an element of *T1* and *U2, V2* is an element of *T2.*

Since *D (script D)* meets the conditions of Lemma 13.2, *D* is a basis for X and Y.

Basis $\mathcal{B} = \{U_1 \times U_2 : U_1 \in \tau_1, U_2 \in \tau_2\}$

Conditions

1: Since $A1 \in \tau_1$ and $A_2 \in \tau_2$, $A_1 \times A_2 \in \mathcal{B}$
are basis elements of T_1 + T_2

2: Suppose $U_1, V_1 \in \tau_1$ and $U_2, V_2 \in \tau_2$
Let X and Y be topological spaces $\Rightarrow T_1$ + T_2 are topologies on T_1 and T_2.
$(U_1 \times U_2) \cap (V_1 \times V_2) = (U_1 \cap V_1) \times (U_2 \cap V_2)$
Let $a \in X$ and let $b \in Y$
Since $(U_1 \cap V_1) \in \tau_1$ containing a and $(U_2 \cap V_2) \in \tau_2$ containing b, we
~~have~~ Let $(U_1 \cap V_1) \times (U_2 \cap V_2) \in \mathcal{B}$
So $(U_1 \times U_2) \cap (V_1 \times V_2)$ is the intersection of one set in $\mathcal{B}$

Projections

It is sometimes useful to express the product topology in terms of a subbasis. To do this, we first define certain functions called projections.

Definition: Let π Pi1 (3.14): *X* x *Y* -> *X* be defined by the equation: Pi1(*x,y*)= *x;*

Let Pi2: *X* x *Y*-> *Y* be defined by the equation:

π Pi2(*x,y*)= *y.*

The maps Pi1 and Pi2 are called the ***projections*** of *X* x *Y* onto its first and second factors, respectively. Pi1 and Pi2 are surjective, unless one of the space *X* or *Y* happens to be empty.

If *U* is an open subset of *X*, then the set Pi1^-1(*U*) is precisely the set *U* x *Y,* which is open in *X* x *Y.* Similarly, if *V* is open in *Y,* then

Pi2^-1(*V*)= *X* x *V,*

which is also open in *X* x *Y.* The intersection of these two sets is the set *U* x *V,* as indicated in the image below. This leads to the following theorem.

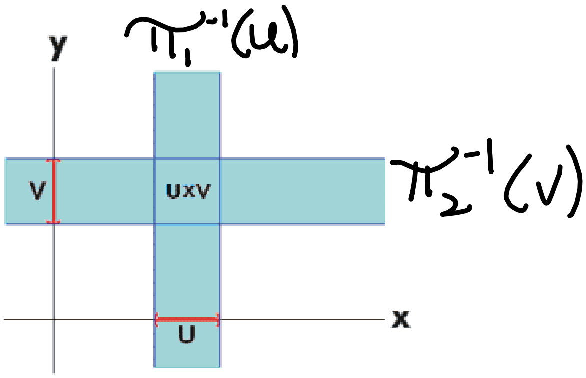


Fig. 8

Theorem 15.2

$$\mathcal{S} = \{\pi_1^{-1}(U) \mid U \text{ open in } X\} \cup \{\pi_2^{-1}(V) \mid V \text{ open in } Y\}.$$

The collection *S (script S)= {Pi1^-1(U)| U open in X} union {Pi2^-1(V)| V open in Y}*

is a subbasis for the product topology on X x *Y.*

Euclidean metric on $\mathbb{R}$

$$d(a,b) = |a-b|$$

Euclidean metric on $\mathbb{R}^2 = \mathbb{R}\times\mathbb{R}$

$$d((x_1,y_1),(x_2,y_2)) = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

. . . $\mathbb{R}^n$

$W = (U \cap Y) \cap A = U \cap A$ $\quad V = U \cap Y$ $\quad U$

$$A \subset Y \subset X$$